GS/ECON 5010 APPLIED MICROECONOMICS

Answers to Midterm Exam October 2005

*Q*1. State and prove Roy's Identity (the relation between a consumer's indirect utility function and her Marshallian demand functions).

*A*1. Roy's Identity :

$$x_i^M(\mathbf{p}, y) = -\frac{\partial v(\mathbf{p}, y)}{\partial p_i} / \frac{\partial v(\mathbf{p}, y)}{\partial y}$$

where $x_i^M(\mathbf{p}, y)$ is the consumer's Marshallian demand function for commodity i, $\mathbf{p}$ is the vector of prices that she faces, y is her income, and $v(\mathbf{p}, y)$ is her indirect utility function.

Proof :

From the first–order conditions for the consumer's utility maximization

$$\frac{\partial u(\mathbf{x})}{\partial x_j} = \lambda p_j$$

where λ is the Lagrange multiplier associated with the consumer's budget constraint.

Since

$$v(\mathbf{p}, y) = u(\mathbf{x}^M(\mathbf{p}, y))$$

therefore

$$\frac{\partial v}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial p_i}$$

which equals

$$\lambda \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i}$$

From the consumer's budget constraint

$$\sum_{j=1}^{n} p_j x_j^M(\mathbf{p}, y) = y$$

Differentiating this budget constraint with respect to p_i yields

$$x_i^M(\mathbf{p}, y) + \sum_{j=1}^{n} p_j \frac{\partial x_j^M}{\partial p_i} = 0$$

which implies that

$$\frac{\partial v}{\partial p_i} = -\lambda x_i^M(\mathbf{p}, y)$$

Also,

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \sum_{j=1}^{n} \frac{\partial x_j^M}{\partial y} \frac{\partial u}{\partial x_j} = \lambda \sum_{j=1}^{n} p_j \frac{\partial x_j^M}{\partial y}$$

Differentiating the consumer's budget constraint with respect to y,

$$\sum_{j=1}^{n} p_j \frac{\partial x_j^M}{\partial y} = 1$$

so that

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \lambda$$

completing the proof of Roy's Identity.

*Q*2. Derive the **Hicksian** (compensated) demand functions of a consumer whose preferences can be represented by the (direct) utility function

$$u(x_1, x_2) = x_1 + \ln x_2$$

*A*2. The most direct way of finding the Hicksian demand functions is to solve the consumer's expenditure minimization problem :

$$\text{minimize}_x \quad \sum_{j=1}^{n} p_j x_j \quad \text{subject} \quad \text{to} \quad u(\mathbf{x}) \geq \bar{u}$$

The first–order conditions for this problem are

$$p_i = \mu \frac{\partial u}{\partial x_i} \quad i = 1, \cdots, n$$

With $n = 2$ and $u(\mathbf{x}) = x_1 + \ln x_2$, these first–order conditions are

$$p_1 = \mu$$

$$p_2 = \frac{\mu}{x_2}$$

so that

$$x_2 = \frac{\mu}{p_2} = \frac{p_1}{p_2}$$

That means that the Hicksian demand for good 2 is

$$x_2^H(\mathbf{p}, u) = \frac{p_1}{p_2}$$

To find the Hicksian demand for good 1, substitute from the utility constraint that $x_1 + \ln x_2 = u$ to get

$$x_1^H(\mathbf{p}, u) = u - \ln x_2 = u - \ln \frac{p_1}{p_2}$$

These results can also be obtained using the "standard" consumer utility maximization problem. Maximizing $u(x_1, x_2)$ subject to the budget constraint $p_1x_1 + p_2x_2 = y$ yields Marshallian demand functions

$$x_1^M(p_1, p_2.y) = \frac{y}{p_1} - 1$$

$$x_2^M(p_1, p_2, y) = \frac{p_1}{p_2}$$

(Note that the Marshallian and Hicksian demand functions for good 2 are identical, since the income elasticity of demand for good 2 is 0.) These Marshallian demand functions imply an indirect utility function of

$$v(p_1, p_2, y) = \frac{y}{p_1} - 1 + \ln\frac{p_1}{p_2}$$

Since $e(\mathbf{p}, v(\mathbf{p}, y)) = y$, therefore

$$e(p_1, p_2, u) = p_1u + p_1 - p_1\ln\frac{p_1}{p_2}$$

Shephard's Lemma says that the Hicksian demand functions are the partial derivatives of the expenditure function with respect to the prices, so that

$$x_1^H(p_1, p_2, y) = u - \ln\frac{p_1}{p_2}$$

$$x_2^H(p_1, p_2, y) = \frac{p_1}{p_2}$$

(These expressions are valid only if $u \geq \ln p_1/p_2$; otherwise $x_1^H(p_1, p_2, y) = 0$ and $x_2^H(p_1, p_2, u) = e^u$.)

*Q*3. Alice and Bob are both risk averse von Neumann–Morgenstern expected utility maximizers. Alice's utility–of–wealth function is

$$u(w) = \ln(w + a)$$

and Bob's utility–of–wealth function is

$$\tilde{u}(w) = \ln(w + b)$$

with

$$b > a > 0$$

(*a*) What are Alice's and Bob's coefficients of relative risk aversion?

(*b*) If Alice is just willing to undertake some risky undertaking, will Bob be willing?

*A*3. The definition of the coefficient of relative risk aversion, $R_R(w)$ is

$$R_R(w) = -\frac{u''(w)w}{u'(w)}$$

When $u(w) = \ln(w + a)$, then

$$u'(w) = \frac{1}{w + a}$$

$$u''(w) = -\frac{1}{(w + a)^2}$$

so that

$$R_R(w) = \frac{w}{w + a}$$

for Alice. Similarly, Bob's coefficient of relative risk aversion is

$$\frac{w}{w + b}$$

If $b > a > 0$, then

$$\frac{w}{w + a} > \frac{w}{w + b}$$

so that Alice is more risk averse than Bob. If the two people had the same initial wealth, then Bob would be willing to undertake any risky investment that Alice is willing to undertake.

(That's not necessarily true if Alice's initial wealth is higher than Bob's. If $a = 1$, $b = 5$, Alice's initial wealth is 1000 and Bob's initial wealth is 10, then Alice is just on the margin of accepting a bet that pays $+2.004$ with probability 1/2, and pays -2.000 with probability 1/2. Bob would prefer his certain wealth of 10 to taking a bet that gives him 2.004 with probability 1/2, and loses him 2 with probability 1/2.)

*Q*4. Show that a firm which is a price taker (on both input and output markets) will make positive economic profits only if its technology exhibits decreasing returns to scale.

*A*4. One way of showing this is to use the definition of the local elasticity of scale

$$\mu(\mathbf{x}) \equiv \frac{1}{f(\mathbf{x})} \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i$$

The firm has decreasing returns to scale if $\mu(\mathbf{x}) < 1$.

Profit maximization by a perfectly competitive firm implies that it chooses input quantities such that

$$p\frac{\partial f(\mathbf{x})}{\partial x_i} = w_i$$

where p is the output price and w_i the price of input i. If $\mu(\mathbf{x}) < 1$, then

$$pf(\mathbf{x}) > p\sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i = \sum_{i=1}^{n} w_i x_i$$

so that profits are positive. (The same argument shows that profits would be non–positive if $\mu(\mathbf{x}) \geq 1$.)

Another demonstration is to consider what would happen if a firm had non–decreasing returns to scale, and could earn positive profits by choosing some vector of inputs $\mathbf{x}$. Let $A > 0$ be the level of those profits. Then choosing an input combination of $2\mathbf{x}$ would earn it profits of at least

$$pf(2\mathbf{x}) - 2\mathbf{w} \cdot \mathbf{x} \geq 2pf(\mathbf{x}) - 2\mathbf{w} \cdot \mathbf{x} = 2A$$

So doubling its inputs would at least double its (positive) level of profit, implying that there is no bound to the profit the firm could make, simply by increasing arbitrarily the scale of its operations.

It also is true that, generally,

$$\frac{\partial^2 C(\mathbf{w}, y)}{\partial y^2} > 0$$

if and only if the firm's technology exhibits decreasing returns to scale at the input combination $x(\mathbf{w}, y)$ chosen to minimize the cost of producing the output level y.

A special case of this result applies if the production technology is homogeneous of some constant degree. If $f(\mathbf{x})$ is homogeneous of degree t, then the cost function can be written

$$C(\mathbf{w}, y) = y^{1/t} C(\mathbf{w}, 1)$$

implying that $\partial^2 C / \partial y^2 > 0$ if and only if $t < 1$. The competitive firm's profit maximization problem will have a unique interior solution only if the second–order condition $-\partial^2 C / \partial y^2 \leq 0$ holds, showing that there is no (interior) solution to the competitive firm's profit maximization problem if its technology is homogeneous of degree $t > 1$.