GS/ECON 5010 3.0A APPLIED MICROECONOMICS : MW section

Answers to Midterm Exam

*Q*1. (*i*) Give an example of preferences which are convex, but not strictly monotonic.

(*ii*) Give an example of preferences which are strictly monotonic but not convex.

*A*1. Of course many valid examples are possible. So here is one simple example of each type.

(*i*) When a person has an "ideal point", and her preferences are determined by (Euclidean) distance from that ideal point, then her preferences will be convex, but not strictly monotonic. So if (t, h) represents a temperature (in degrees Celsius) and a relative humidity level (in percentage), the person might find one particlular temperature–humidity combination (t^*, h^*) to be the absolute best. Asked to rank two other combinations (t, h) and (t', h'), she would rank higher the bundle which is closest to her ideal combination (t^*, h^*). These preferences could be represented by a utility function

$$u(t, h) = -(t - t^*)^2 - (h - h^*)^2$$

These preferences are not strictly monotonic : if $(t^*, h^*) = (20, 30)$ then she would prefer $(22, 35)$ to $(25, 50)$. But they are convex, since the "at least as good as" sets are circular disks (centred at (t^*, h^*)), which are convex sets.

(*ii*) If a person's preferences were represented by the utility function

$$U(x, y) = x^2 + y^2$$

then her preferences are strictly monotonic, since the function is strictly increasing in both arguments. However, the indifference curves for the above utility function have the equation

$$y = \sqrt{\bar{u} - x^2}$$

for some reference level of utility $\bar{u}$, and $y = \sqrt{\bar{u} - x^2}$ defines a curve which gets more steep as x (the variable measured on the horizontal axis) gets larger.

An alternate proof that these preferences are not convex : $U(6, 0) = U(0, 6) = 36$. But the utility from a point halfway between $(6, 0)$ and $(0, 6)$, $U(3, 3) = 18 < 36$. A single counter–example such as this one is enough to show the preferences are not convex.

*Q*2. If a person's expenditure function is

$$e(\mathbf{p}, u) = [(p_1)^{1/3} + (p_2)^{1/3} + (p_3)^{1/3}]^a u$$

(*i*) What is a?

(ii) What is her Hicksian demand function for good 1?

(iii) What is her indirect utility function?

$A2$. An expenditure function must be homogeneous of degree 1 in all prices. If we multiply all prices by k, then

$$e(k\mathbf{p},u) = [(kp_1)^{1/3} + (kp_2)^{1/3} + (kp_3)^{1/3}]^a u = (k)^{a/3}[(p_1)^{1/3} + (p_2)^{1/3} + (p_3)^{1/3}]^a u$$

Therefore $e(k\mathbf{p},u) = ke(\mathbf{p},u)$ if and only if $a = 3$; the answer to part (i) is $a = 3$ because only then will the expenditure function be homogeneous of degree 1.

To find the Hicksian demands, use Shephard's Lemma : $x_i^h(\mathbf{p},u) = \frac{\partial e(\mathbf{p},u)}{\partial p_i}$. Given the expenditure function,

$$x_i^h(\mathbf{p},u) = \frac{a}{3}[(p_1)^{1/3} + (p_2)^{1/3} + (p_3)^{1/3}]^{a-1}(p_i)^{-2/3}u$$

Since $a = 3$, therefore

$$x_i^h = [(p_1)^{1/3} + (p_2)^{1/3} + (p_3)^{1/3}]^2(p_i)^{-2/3}u$$

(As a check, note that if each x_i^h is defined in this way, then

$$\begin{aligned} p_1x_1^h + p_2x_2^h + p_3x_3^h &= [(p_1)^{1/3} + (p_2)^{1/3} + (p_3)^{1/3}]^2[p_1(p_1)^{-2/3} + p_2(p_2)^{-2/3} + p_3(p_3)^{-2/3}] \\ &= [(p_1)^{1/3} + (p_2)^{1/3} + (p_3)^{1/3}]^3 u = e(\mathbf{p},u) \end{aligned}$$

as is required.)

Since $e[\mathbf{p}, v(\mathbf{p},y)] = y$, here

$$[(p_1)^{1/3} + (p_2)^{1/3} + (p_3)^{1/3}]^3 v(\mathbf{p},y) = y$$

so that

$$v(\mathbf{p},y) = [(p_1)^{1/3} + (p_2)^{1/3} + (p_3)^{1/3}]^{-3}y$$

is the indirect utility function.

$Q3$. An expected utility maximizer has utility–of–wealth function

$$U(W) = 200 - \frac{1}{W}$$

She has initial wealth of 1 million dollars. An opportunity arises to invest half her wealth in a stock. With probability π the stock will triple in value (to \$1.5 million) ; with probability $1 - \pi$ the stock will be worthless.

What must π be in order for this person to be willing to make this investment?

$A3$. This is a special case of question #5 of assignment 2.

There are two states of the world here, if she undertakes the investment. In the good state, which occurs with probability π, her investment triples, giving her a total wealth of $W/2+3W/2 = 2W$. In the bad state, which occurs with probability $1-\pi$, her investment is worthless, leaving her with only the half of her wealth $W/2$ which she did not invest.

Therefore, her expected utility if she undertakes the investment is

$$EU = \pi[200 - \frac{1}{2W}] + (1-\pi)[200 - \frac{1}{(W/2)}] \qquad (3-1)$$

If she does not undertake the investment, her utility is simply the utility of her initial wealth

$$U = 200 - \frac{1}{W} \qquad (3-2)$$

If she is indifferent between undertaking the investment and not undertaking it, expression $(3-1)$ must equal expression $(3-2)$, or

$$\pi\frac{1}{2W} + (1-\pi)\frac{1}{W/2} = \frac{1}{W} \qquad (3-3)$$

Multiplying both sides of equation $(3-3)$ by $2W$,

$$\pi + 4(1-\pi) = 2 \qquad (3-4)$$

or

$$\pi = \frac{2}{3} \qquad (3-5)$$

With these preferences, she will undertake the investment if and only if the probability π of success is $2/3$ or more.

(Due to the fact that her preferences exhibit constant relative risk aversion, the level of her initial wealth does not affect the value of this threshold probability.)

*Q*4. Could a firm's unconditional demand function for an input slope up? Explain your anwwer.

*A*4. The answer is "no", and the explanation follows from the convexity of the profit function, and from Hotelling's lemma.

Hotelling's lemma says defines the relation between the firm's unconditional demand function for an input, and its profit function

$$x_i^u(p, \mathbf{w}) = \frac{\partial \pi(p, \mathbf{w})}{\partial w_i} \qquad (4-1)$$

The convexity of the firm's profit function means that the matrix H must be positive semi–definite, where H is the matrix of second derivatives of the profit function.

A necessary condition for a matrix to be positive semi–definite is for the elements along the diagonal of the matrix to be non–negative. Therefore it must be true that

$$H_{ii} \geq 0 \quad i = 1, 2, \cdots, n+1 \tag{4-2}$$

But Hotelling's lemma implies that the elements along the diagonal of H are the negatives of the own–price derivatives of the unconditional input demands : differentiation of $(4-1)$ with respect to i says that

$$\frac{\partial x_i^u(p, \mathbf{w})}{\partial w_i} = -\frac{\partial^2 \pi}{\partial w_i^2} = -H_{i+1,i+1} \leq 0 \tag{4-3}$$

which shows the unconditional demand curves for inputs cannot slope up. (The reason for the $i+1$ in the last part of equation $(4-3)$ is that the first row and column of H contain derivatives with respect to the output price p, so that derivatives with respect to the input prices start at row 2 or column 2.)