GS/ECON 5010 3.0B APPLIED MICROECONOMICS : Friday section

Answers to Midterm Exam October 2008

*Q*1. Prove Roy's Identity (the theorem relating Marshallian demand functions and the indirect utility function).

*A*1. Roy's Identity is

$$\frac{\partial v(\mathbf{p},y)}{\partial p_i} = -\frac{\partial v(\mathbf{p},y)}{\partial y} x_i(\mathbf{p},y) \tag{1-1}$$

where $v(\mathbf{p},y)$ is the indirect utility function, $\mathbf{p}$ the vector of prices faced by the consumer, y the income the consumer has to spend, and $x_i(\mathbf{p},y)$ the consumer's Marshallian demand function for good i.

Theorem 1.6 in *Jehle and Reny* provides a proof of Roy's Identity, using the Envelope Theorem. An alternative (longer) proof uses the following two properties :

$$p_1 x_1(\mathbf{p},y) + p_2 x_2(\mathbf{p},y) + \cdots p_n x_n(\mathbf{p},y) = y \tag{1-2}$$

from the consumer's budget constraint, and

$$\frac{\partial u(\mathbf{x})}{\partial x_i} = \lambda p_i \tag{1-3}$$

at the consumer's optimum, where $u(\mathbf{x})$ is the consumer's direct utility function, and λ is the multiplier on the consumer's budget constraint in her maximization problem (of maximizing $u(\mathbf{x})$ with resspect to $\mathbf{x}$ subject to the budget constraint $\mathbf{p} \cdot \mathbf{x} = y$).

By definition

$$v(\mathbf{p},y) = u(x_1(\mathbf{p},y), x_2(\mathbf{p},y), \ldots, x_n(\mathbf{p},y)) \tag{1-4}$$

Differentiating $(1-4)$ with respect to p_i yields

$$\frac{\partial v(\mathbf{p},y)}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j(\mathbf{p},y)}{\partial p_i} \tag{1-5}$$

Substituting from $(1-3)$ into $(1-5)$

$$\frac{\partial v(\mathbf{p},y)}{\partial p_i} = \lambda \sum_{j=1}^{n} p_j \frac{\partial x_j(\mathbf{p},y)}{\partial p_i} \qquad (1-6)$$

Differentiating the budget constarint $(1-2)$ with respect to p_i yields

$$x_i(\mathbf{p},y) + \sum_{j=1}^{n} p_j \frac{\partial x_j(\mathbf{p},y)}{\partial p_i} = 0 \qquad (1-7)$$

Substitution of $(1-7)$ into $(1-6)$ implies that

$$\frac{\partial v(\mathbf{p},y)}{\partial p_i} = -\lambda x_i(\mathbf{p},y) \qquad (1-8)$$

Next, differentiating the definition $(1-4)$ with respect to y implies that

$$\frac{\partial v(\mathbf{p},y)}{\partial y} = \sum_{j=1}^{n} \frac{\partial x_j(\mathbf{p},y)}{\partial y} \qquad (1-9)$$

Differentiation of the budget constraint $(1-2)$ with respect to y implies that

$$\sum_{j=1}^{n} p_j \frac{\partial x_j(\mathbf{p},y)}{\partial y} = 1 \qquad (1-10)$$

Substitution of $(1-3)$ and $(1-10)$ into $(1-9)$ therefore implies that

$$\frac{\partial v(\mathbf{p},y)}{\partial y} = \lambda \qquad (1-11)$$

which means that $(1-8)$ is the same as $(1-1)$, completing the (long version of the) proof of Roy's Identity.

*Q*2. Is it possible that the following data represent the behaviour of a consumer with well–behaved preferences? In the table, p_i^t is the price of good i in year t and x_i^t is the quantity consumed of good i in year t.

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	1	1	1	10	2	8
2	3	1	3	5	12	4
3	1	2	1	8	1	10
4	1	1	3	8	6	7

*A*2. If the costs of the different bundles in the different years are arranged in a matrix (in which the element in the i–th column of the j–th row is the cost of bundle $\mathbf{x}^i$ in year j), the costs are

t	$\mathbf{x}^1$	$\mathbf{x}^2$	$\mathbf{x}^3$	$\mathbf{x}^4$
1	20	21	19	21
2	56	39	55	51
3	22	33	20	27
4	36	29	39	35

Bundle $\mathbf{x}^i$ is revealed directly to be preferred to bundle $\mathbf{x}^j$ if : in year i, bundle $\mathbf{x}^j$ cost less than (or the same amount as) bundle $\mathbf{x}^i$. That is, $\mathbf{x}^i$ d.r.p.t. $\mathbf{x}^j$ if and only if $C_{ii} \geq C_{ij}$, where the C_{ij}'s are elements in the cost matrix above.

From that matrix, the only cases of one bundle being revealed referred to another occur in years 1 and 4 ; in years 2 and 3, the bundle actually chosen is cheaper than any of the other three bundles, so that we cannot tell if she chose bundles $\mathbf{x}^2$ and $\mathbf{x}^3$ because she liked them more than the other bundles, or because they were cheaper.

In year 1, $C_{11} > C_{13}$ so that year 1 data show that $\mathbf{x}^1$ d.r.p.t. $\mathbf{3}^4$. In year 4, $C_{44} > C_{42}$, so that $\mathbf{x}^4$ d.r.p.t. $\mathbf{x}^2$.

Therefore, the person's behaviour reveals no violations of WARP or of SARP. The data could represent the choices of a consumer with consistent, well–behaved preferences.

*Q*3. An expected utility maximizer has utility–of–wealth function

$$U(W) = 200 - \frac{1}{W}$$

Calculate this person's risk premium for a gamble which offers a wealth of $2X$ with probability 0.5, and of a wealth of $(0.5)X$ with probability 0.5, where X is some positive number.

*A*3. To calculate the risk premium for a gamble, first the certainty equivalent to the gamble must be calculated. For any gamble g, calculate the expected utility of the gamble :

the certainty equivalent is the certain amount of wealth CE which offers the same expected utility. That is, if $g = (p_1 \circ W_1, p_2 \circ W_2)$ then CE is the solution to

$$U(CE) = p_1 U(W_1) + p_2 U(W_2) \tag{3 – 1}$$

Here, $(3-1)$ becomes

$$200 - \frac{1}{CE} = (0.5)(200 - \frac{1}{2X}) + (0.5)(200 - \frac{1}{(0.5)X}) \tag{3 – 2}$$

Equation $(3-2)$ simplifies to

$$\frac{1}{CE} = \frac{1}{2}\frac{1}{2X} + \frac{1}{2}\frac{2}{X} \tag{3 – 3}$$

or

$$CE = \frac{4}{5}X \tag{3 – 4}$$

The risk premium for the gamble is the difference between the expected value $Eg = p_1 W_1 + p_2 W_2$ of the gamble, and the certainty equivalent to the gamble. Here

$$Eg = (0.5)(2X) + (0.5)((0.5)X) = \frac{5}{4}X \tag{3 – 5}$$

so that the risk premium equals

$$Eg - CE = (\frac{5}{4} - \frac{4}{5})X = \frac{9}{20}X \tag{3 – 6}$$

*Q*4. What is the cost function $C(\mathbf{w}, y)$ for a firm for which the production function is

$$f(x_1, x_2) = \ln(x_1 + 1) + x_2$$

where x_i is the quantity employed of input i?

*A*4. The cost function is the cost of the input bundle $(x_1(\mathbf{w}, y), x_2(\mathbf{w}, y))$ which minimizes the cost $w_1 x_1 + w_2 x_2$ subject to the constraint that $f(x_1, x_2) = y$.

The first–order conditions for the above minimization problem are

$$\mu\frac{\partial f(x_1,x_2)}{\partial x_i} = w_i \quad i = 1,2 \qquad (4-1)$$

where μ is the Lagrange multiplier on the constraint $f(x_1,x_2) = y$. For the given production function, the partial derivatives are $f_1 = \frac{1}{x_1+1}$ and $f_2 = 1$ so that $(4-1)$ becomes

$$\frac{\mu}{x_1+1} = w_1 \qquad (4-2)$$

$$\mu = w_2 \qquad (4-3)$$

Substituting from $(4-3)$ into $(4-2)$,

$$\frac{w_1}{x_1+1} = w_2 \qquad (4-4)$$

or

$$x_1(\mathbf{w},y) = \frac{w_2}{w_1} - 1 \qquad (4-5)$$

which is the conditional input demand for input 1. Since $\ln(x_1+1) + x_2 = y$, therefore

$$x_2 = y - \ln(x_1+1) \qquad (4-6)$$

implying a conditional demand function for input 2 of

$$x_2(\mathbf{w},y) = y - \ln(\frac{w_2}{w_1}) = y - \ln w_2 + \ln w_1 \qquad (4-7)$$

Since the cost function $C(\mathbf{w},y)$ is the cost of the inputs,

$$C(\mathbf{w},y) = w_1x_1(\mathbf{w},y) + w_2x_2(\mathbf{w},y) = w_2 - w_1 + w_2y - w_2\ln w_2 + w_2\ln w_1 \qquad (4-8)$$

The right–side expression in equation $(4-8)$ is the firm's cost function : partial differentiation of $w_1 - w_2 + w_2y - w_2\ln w_2 - w_2\ln w_1$ with respect to w_1 and w_2 yields the right sides of equations $(4-5)$ and $(4-7)$ respectively, confirming Shephard's Lemma.

However, expression $(4-5)$ makes sense only if $w_2 \geq w_1$. If $w_2 < w_1$, then $MP_1/MP_2 < w_1/w_2$, even if $x_1 = 0$. In this case, the firm uses only input 2. So if $w_1 > w_2$, then

$$x_1(\mathbf{w}, y) = 0 \tag{4-9}$$

$$x_2(\mathbf{w}, y) = y \tag{4-10}$$

and

$$C(\mathbf{w}, y) = w_2 y \tag{4-11}$$