YORK UNIVERSITY Faculty of Graduate Studies

Final Examination December 14, 2010

Economics 5010 AF3.0 : Applied Microeconomics S. Bucovetsky

time=2.5 hours

Do any **6** of the following 10 questions. All count equally.

*Q*1. If a person's preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = x_1 + 100 - \frac{1}{x_2} + \ln x_3$$

find the person's Marshallian demand functions for each good, her indirect utility function, her Hicksian demand functions, and her expenditure function.

*A*1. Start with the Marshallian demand functions, which are the solutions to the "standard" consumer optimization problem, of maximizing $u(\mathbf{x})$ subject to the constraint that $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{y}$. The first–order conditions for this maximization are

$$u_1 = 1 = \lambda p_1 \qquad (1-1)$$

$$u_2 = \frac{1}{(x_2)^2} = \lambda p_2$$

$$u_3 = \frac{1}{x_3} = \lambda p_3 \qquad (1-4)$$

(where λ is the Lagrange multiplier on the budget constraint), as well as the budget constraint

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = y \qquad (1-4)$$

The quasi–linearity of the utility function means that the Marshallian demand functions for goods 2 and 3 can be derived without even using the budget constraint : since $(1-1)$ implies that $\lambda = 1/p_1$, $(1-2)$ and $(1-3)$ can be written

$$x_2 = \sqrt{\frac{p_1}{p_2}} \qquad (1-5)$$

$$x_3 = \frac{p_1}{p_3} \tag{1-6}$$

which are the Marshallian demand functions for goods 2 and 3. Substituting from $(1-5)$ and $(1-6)$ into the budget constraint,

$$x_1^M = \frac{y - p_2 x_2 - p_3 x_3}{p_1} = \frac{y}{p_1} - \sqrt{\frac{p_2}{p_1}} - 1 \tag{1-7}$$

is the Marshallian demand for good 1 (provided that income is high enough that $y > \sqrt{p_1 p_2} + p_1$, which ensures that $x_1 > 0$).

Substituting from $(1-5)$–$(1-7)$ into the utility function, the indirect utility function is

$$v(\mathbf{p}, y) = \frac{y}{p_1} - \sqrt{\frac{p_2}{p_1}} - 1 + 100 - \sqrt{\frac{p_2}{p_1}} + \ln p_1 - \ln p_3 = \frac{y}{p_1} - 2\sqrt{\frac{p_2}{p_1}} + 99 + \ln p_1 - \ln p_3 \tag{1-8}$$

To find the Hicksian demands, minimization of $\mathbf{p} \cdot \mathbf{x}$ subject to $x_1 + 100 - \frac{1}{x_2} + \ln x_3 \geq u$, yields first–order conditions

$$p_1 = \mu u_1 = \mu \tag{1-9}$$

$$p_2 = \mu u_2 = \mu \frac{1}{(x_2)^2} \tag{1-10}$$

$$p_3 = \mu u_3 = \mu \frac{1}{x_3} \tag{1-11}$$

(where μ is the Lagrange multiplier on the utility constraint) as well as the utility constraint

$$x_1 + 100 - \frac{1}{x_2} + \ln x_3 = u \tag{1-12}$$

Again, quasi–linearity helps a lot : substituting $\mu = p_1$ from $(1-9)$ into $(1-10)$ and $(1-11)$ yields the Hicksian demand functions for goods 2 and 3,

$$x_2 = \sqrt{\frac{p_1}{p_2}} \tag{1-13}$$

$$x_3 = \frac{p_1}{p_3} \tag{1-14}$$

which are the same as the Marshallian demand functions (as the Slutsky equation requires). The Hicksian demand function can be obtained from substitution of $(1-13)$ and $(1-14)$ into the utility constraint $(1-12)$,

$$x_1^H = u - 100 + \frac{1}{x_2} - \ln x_3 = u - 100 + \sqrt{\frac{p_2}{p_1}} - \ln p_1 + \ln p_3 \qquad (1-15)$$

Then the expenditure function is

$$e(\mathbf{p}, u) = p_1 x_1^H + p_2 x_2^H + p_3 x_3^H = p_1 u - 100 p_1 + \sqrt{p_2 p_1} - p_1 \ln p_1 + p_1 \ln p_3 + \sqrt{p_1 p_2} + p_1 = p_1 u - 99 p_1 - p_1 \ln \qquad (1-16)$$

[Alternatively, the expenditure function $(1-16)$ can be obtained from the duality relation $v(\mathbf{p}, e(\mathbf{p}, u)) = u$ and the indirect utility function $(1-8)$, and then the Hicksian demand functions obtained from differentiation of $(1-16)$ with respect to the prices.]

*Q*2. Person A is a risk–averse expected utility maximizer, with utility–of–wealth function $u(W)$. Person B is also an expected utility maximizer, with utility–of–wealth function $V(W) = f[u(W)]$ where $f(\cdot)$ is an increasing concave function, and where $u(\cdot)$ is person A's utility–of–wealth function.

Show that person B has a higher risk premium for any risky gamble than person A.

*A*2. A couple of ways of doing this :

(i) Person B having a higher risk premium for any risky gamble than person A is equivalent to person B having a higher coefficient of risk aversion than person A.

Person A's coefficient of relative risk aversion is

$$R_R^A \equiv -\frac{u''(W)W}{u'(W)}$$

Person B's coefficient of relative risk aversion is

$$R_R^B = -\frac{U''(W)W}{U'(W)}$$

But the chain rule implies that

$$U'(W) = f'[u(W)]u'(W) \tag{2-1}$$

so that

$$U''(W) = f''[u(W)][u'(W)]^2 + f'[u(W)]u''(W) \tag{2-2}$$

meaning that

$$R_R^B = -\frac{f''[u(W)]u'(W)W}{f'[u(W)]} - \frac{u''(W)W}{u'(W)} = -\frac{f''u'W}{f'} + R_R^A \tag{2-3}$$

Since $f'' < 0$ and $f' > 0$, equation $(2-3)$ implies that person B has a higher coefficient of relative risk aversion than person A.

(ii) Alternatively, the certainty equivalent to some gamble for person A is defined by

$$E[u(W)] = u(CE^A) \tag{2-4}$$

Now if the function $f(\cdot)$ is concave, then

$$f(E[u(W)]) \geq E[f(u(W)] \tag{2-5}$$

for any gamble (which is not a sure thing)* so that

$$E[U(W)] \equiv E[f(u(W))] \leq f(E[u(W)]) = f[u(CE^A)] = U(CE^A) \tag{2-6}$$

Since equation $(2-6)$ says that $U(CE^A) \geq E[U(W)]$, therefore, person B's certainty equivalent CE^B to the gamble must be no greater than CE^A.

And if person B always has a lower certainty equivalent to any gamble (than person A), then he must have a higher risk premium.

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* Why? The definition of concavity says that $f(tx+(1-t)y) > tf(x)+(1-t)f(y)$ if $f(\cdot)$ is concave. That means that $f(\sum \pi_i W_i) \geq \sum \pi_i f(W_i)$ for any probabilities $(\pi_1, \ldots, \pi_n)$ of n events.

$Q3$. What is a firm's cost function, if its production function is

$$y = (\sqrt{x_1} + \sqrt{x_2})^3$$

where y is the quantity of output, and x_1 and x_2 are the quantities used of two inputs?

$A3$. This is an example of a CES production function,

$$f(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\mu/\rho}$$

with $\rho = 0.5$ and $\mu = 1.5$, so that it's a slight modification [to the case in which $\mu > 1$] of example 3.3 (pp. 128–129) in Jehle and Reny.

Minimization of $w_1x_1 + w_2x_2$ subject to the constraint $f(x_1, x_2) = (\sqrt{x_1} + \sqrt{x_2})^3 \geq y$ implies first–order conditions

$$w_1 = \mu\frac{3}{2}(\sqrt{x_1} + \sqrt{x_2})^2[x_1]^{-0.5} \quad (3-1)$$

$$w_2 = \mu\frac{3}{2}(\sqrt{x_1} + \sqrt{x_2})^2[x_2]^{-0.5} \quad (3-2)$$

where μ is the Lagrange multiplier on the constraint. So $(3-1)$ and $(3-2)$ imply that

$$x_2 = [\frac{w_1}{w_2}]^2 x_1 \quad (3-3)$$

Substituting for x_2 from $(3-3)$ into the constraint,

$$y = (\sqrt{x_1} + \frac{w_1}{w_2}\sqrt{x_1})^3 = [\frac{w_1 + w_2}{w_2}]^3 x_1^{3/2} \quad (3-4)$$

which is the conditional input demand for input 1, since it can be re–written

$$x_1 = [\frac{w_2}{w_1 + w_2}]^2 y^{2/3} \quad (3-5)$$

Substituting back into $(3-3)$, then

$$x_2 = [\frac{w_1}{w_1 + w_2}]^2 y^{2/3} \quad (3-6)$$

is the conditional demand for input 2. Therefore, the cost function is

$$w_1x_1 + w_2x_2 = w_1[\frac{w_2}{w_1+w_2}]^2 y^{2/3} + w_2[\frac{w_1}{w_1+w_2}]^2 y^{2/3}$$

or

$$C(w_1, w_2, y) = \frac{w_1w_2}{w_1+w_2} y^{2/3} \qquad (3-7)$$

*Q*4. What is the equation of the long–run industry supply curve of a perfectly competitive industry in which there are a large number of identical firms, each of which has the same total cost function

$$TC(y) = 2y^3 - 48y^2 + 388y$$

where $TC(y)$ is the total cost of producing y units of output?

*A*4. The key here is that the firms are identical, and that there are a lot of them. In long–run equilibrium, each firm must break even (otherwise there would be entry or exit).

Breaking even means each firm's price must equal its average costs (that's the definition of zero profits). Since each firm also chooses to maximize profits by finding an output level at which $p = MC$, it must be the case that $MC = AC$ in any long–run equilibrium.

Since the average cost function associated with the total cost function $TC(y) = 2y^3 - 48y^2 + 388y$ is U–shaped, there is a unique output level for each firm at which $MC = AC$.

The problem is then to find the output level y^* for which $MC = AC$, and the level of $MC = AC$ when $y = y^*$. The industry long–run supply curve is a horizontal line, at a height of $AC(y^*)$.

When $TC(y) = 2y^3 - 48y^2 + 388y$, then

$$AC(y) \equiv TC(y)/y = 2y^2 - 48y + 388 \qquad (4-1)$$

so that

$$AC'(y) = 4y - 48 \qquad (4-2)$$

(so that the average cost curve is indeed U–shaped) implying that $AC(y)$ is minimized at the y^* for which $4y^* - 48 = 0$, or

$$y^* = 12$$

Evaluating the average cost at $y = 12$,

$$AC(y^*) = 2(12^2) - 48(12) + 388 = 100 \qquad (4-3)$$

so that the industry long–run supply curve is a horizontal line, at a price of 100.

*Q*5. How does the equilibrium price vary with the number of firms n, in the following n–firm model of oligopoly?

Firms produce an identical good. Each firm has the same constant–returns–to–scale technology, so that the total cost of producing y units of the good is cy for any firm. Firms choose their prices simultaneously and non–cooperatively, and buyers buy from the lowest–cost firm. (If 2 or more firms are tied with the lowest price, they split the market evenly.)

*A*5. Short and (I hope) sweet). It's a Bertrand model ("Firms choose their prices simultaneously and non–cooperatively") with identical, constant marginal costs.

So the unique equilibrium to the n–firm Bertrand game is for price to be driven down to marginal cost c. The equilibrium price does not vary with the number of firms in the industry (as long as there are at least 2 of them).

*Q*6. What are all the Pareto efficient allocations in the following 3–person exchange economy?

There are 2 goods : the aggregate endowment of good 1 is 30 units, and the aggregate endowment of good 2 is 40 units.

Person 1's preferences can be represented by the utility function $U^1(x_1^1, x_2^1) = x_1^1 + x_2^1$.

Person 2's preferences can be represented by the utility function $U^2(x_1^2, x_2^2) = x_1^2 x_2^2$.

Person 3's preferences can be represented by the utility function

$U^3(x_1^3, x_2^3) = \min(x_1^3, x_2^3)$.

*A*6. Consider first "interior" allocations, that is allocations in which each person's consumption of each good is positive. For such an allocation, the condition for Pareto efficiency is that the MRS between the 2 goods be the same for all three people. Since person 1 regards the 2 goods as perfect substitutes, her MRS is 1, regardless of how much she consumes of either good. Therefore, in any interior Pareto optimum, the MRS of all 3 people must equal 1. That means (naturally) that person 3 consumes equal quantities of both goods : when $x_1^3 = x_2^3$, MRS^3 is undefined, but only at these "kinks" in her indifference curve could her MRS be between 0 and ∞. For person 2 to have an MRS of 1, it must be the case that $x_1^2 = x_2^2$, since $MRS^2 = x_2^2/x_1^2$.

So the Pareto efficient "interior" allocations are those for which $x_1^2 = x_2^2 > 0$, $x_1^3 = x_3^3 > 0$ and $x_1^2 + x_1^3 < 30$.

What about "corner" solutions, in which some person's consumption of some good is zero? Since person #3 regards the two goods as strict complements, it must be the case that $x_1^3 = x_2^3$ in **any** Pareto efficient allocation. (If $x_1^3 > x_2^3$, for example, reallocating the excess of good 1 to person #1 would make person #1 better off without making person # any worse off.)

It also can't be efficient to have $x_1^2 > x_2^2$. If this were true, then it would have to be true that person #1 was consuming some of good 2. (There is more of good 2 in aggregate than thee is of good 1, and person #3 never consumes any of the extra good 2.) And then having person #2 trade some of his good 1 to person 2 for some of good 2 would be Pareto–improving.

However, if may be efficient to have $x_2^2 > x_1^2$, so long as $x_1^1 = 0$.) Any allocation in which $x_2^2 > x_1^2$, and $x_1^1 = 0$ will be Pareto efficient, provided that $x_1^3 = x_2^3$ (and provided that $x_1^1 + x_1^2 + x_1^3 = 30$ and $x_2^1 + x_2^2 + x_2^3 = 40$).[For example, $x_1^1 = 0, x_2^1 = 6, x_1^2 = 10, x_2^2 =$

$14, x_1^3 = 20, x_2^3 = 20$ is efficient.]

In summary, the Pareto efficient allocations are those for which either

i $x_1^2 = x_2^2 > 0$, $x_1^3 = x_3^3 > 0$ and $x_1^2 + x_1^3 < 30$

or

ii $x_1^1 = 0$, $x_2^2 \geq x_1^2$, $x_1^3 = x_2^3$

*Q*7. Prove (both)

(i) that every Walrasian (competitive) equilibrium allocation is in the core

(ii) that every allocation in the core is Pareto optimal

*A*7 The first part is Theorem 5.6 in Jehle and Reny.

It must be shown that if $\mathbf{x}$ as a competitive equilibrium allocation, then it cannot be blocked by any coalition S.

So let $\mathbf{p}$ be a vector of equilibrium prices, and $\mathbf{x}$ the associated competitive equilibrium allocation.

For every person i, her consumption bundle $\mathbf{x}^i$ is the one she chooses, when prices are $\mathbf{p}$, and when she has income equal to the value of her endowment $\mathbf{p} \cdot \mathbf{e}^i$. Since her consumption bundle is on her budget line, it must be true that

$$\mathbf{p} \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{e}^i \qquad (7-1)$$

She prefers $\mathbf{x}^i$ to any other consumption bundle in her budget set. Therefore, if there is some other bundle $\mathbf{y}^i$ which she prefers (weakly) to $\mathbf{x}^i$, then it must be the case that $\mathbf{y}^i$ costs more than $\mathbf{x}^i$ at prices $\mathbf{p}$, if she prefers $\mathbf{y}^i$ weakly to $\mathbf{x}^i$:

$$\mathbf{p} \cdot \mathbf{y}^i > \mathbf{p} \cdot \mathbf{x}^i \qquad (7-2)$$

(whenever i prefers $\mathbf{y}^i$ weakly to $\mathbf{x}^i$).

So suppose that some coalition S tries to block the competitive equilibrium allocation $\mathbf{x}$ with some other allocation $\mathbf{y}$. Since each member i of S must like $\mathbf{y}^i$ at least as much as $\mathbf{x}^i$, and at least one member must prefer $\mathbf{y}^i$ strictly, equations $(7-1)$ and $(7-2)$ imply that $\mathbf{p} \cdot \mathbf{y}^i \geq \mathbf{p} \cdot \mathbf{e}^i$ for each member of the coalition, with the inequality strict for at least one member. Adding up over all members of the potential blocking coalition S,

$$\sum_{i \in S} \mathbf{p} \cdot \mathbf{y}^i > \sum_{i \in S} \mathbf{p} \cdot \mathbf{e}^i \qquad (7-3)$$

But the proposed allocation for the coalition S must be feasible in order for S to block $\mathbf{x}$ with $\mathbf{y}$: the allocation must come from the endowments of the members of S, so that

$$\sum_{i \in S} y_j^i \leq \sum_{i \in S} e_j^i \qquad (7-4)$$

for each good j.

Since each price p_j is non–negative, equations $(7-3)$ and $(7-4)$ **cannot** both hold. Therefore, it is impossible for any coalition S to block the competitive equilibrium allocation $\mathbf{x}$, so that $\mathbf{x}$ is in the core.

The second part of the question is a simple observation from the definition of the core (see the second paragraph after the definition of "blocking coalitions" on page 186 in Jehle and Reny). If an allocation is in the core, then it cannot be blocked by the "grand coalition" consisting of all people in the economy. So if an allocation is in the core, there is no other feasible allocation which is preferred weakly by all people in the economy, and strictly by at least one person. So any core allocation must be Pareto optimal.

$Q8$. What are all the Nash equilibria (in pure and mixed strategies) to the following game in strategic form?

	a	b	c	d
I	$(4,2)$	$(3,1)$	$(2,0)$	$(2,0)$
II	$(2,3)$	$(0,0)$	$(2,3)$	$(1,2)$
III	$(0,2)$	$(4,1)$	$(0,1)$	$(2,4)$
IV	$(0,4)$	$(10,2)$	$(0,2)$	$(1,1)$

*A*8. There are several strategies which can be removed using iterated elimination of weakly dominated strategies. This removal is useful in trying to find any mixed–strategy : a weakly dominated strategy (or a strategy which is weakly dominated once other weakly dominated strategies have been eliminated) cannot be played with positive probability in a purely mixed strategy.

Column b is a strictly dominated (by column a) strategy for player 2. Column c is weakly dominated by column a, for player 2, as well. Row II is weakly dominated by row I for player 1. And with b and c eliminated, rows III and IV are both weakly dominated by row I for player I.

So that leaves only row I as undominated for player 1. And column a is player 2's best response to I.

So the game can be solved by iterated elimination of weakly dominated strategies. And (I, a) is the pair of pure strategies in that solution.

But (I, a) is not the only Nash equilibrium in pure strategies. Both (II, c) and (III, d) are also Nash equilibria (even though they require players to play weakly dominated strategies).

Those are the three Nash equilibria in pure strategies : (I, a), (II, c) and (III, d).

And there are no other equilibria, since neither player will ever put positive weight on a weakly dominated strategy in a mixed–strategy Nash equilibrium.

*Q*9. If there are three bidders in an auction, and each bidder's private value of the object being auctioned is an independent draw from the set of values $\{1, 2, 3\}$, with each of the 3 values equally likely,

i What is the expected revenue from an auction which allocates the object efficiently?

ii Design an auction which has a higher expected revenue than the efficient auction.

*A*9. The easiest efficient auction to use here is probably the second–price sealed–bid auction (or equivalently, the English ascending–bid oral auction), since bidders have a dominant strategy in this auction, to bid their true private values.

If each bidder bids her value, then the expected revenue from this auction is the expected value of the second–highest of the three bids.

With 3 bidders, and 3 possible values for each bidder, there are $3^3 = 27$ possible outcomes. The second highest bid is 3 in 7 of those outcomes $((3,3,3),(3,3,2),(3,3,1),(3,1,3),(3,2,3),(1,3,3),(2,3,3))$, and it is 1 in another 7 of the outcomes $((1,1,1),(1,1,3),(1,1,2),(1,3,1),(1,2,1),(3,1,1),(2,1,1))$. In the remaining 13 outcomes, the second–highest bid is 2. So the expected value of the second–highest bid is $(7/27)(3) + (7/27)(1) + (13/81)(2) = 2$.

But with independent private values, every efficient auction has the same expected revenue, so that 2 is the expected revenue from any efficient auction here.

If the auctioneer put in a reserve price of 1.5, then the reserve bid would not be met if all 3 bidders valued the object at 1. There's a 1/27 chance of that happening. If the second–highest value is 1, but the highest value is 2 or 3, then the presence of the reserve bid raises the price from 1 (the second–highest bid) to 1.5 (the reserve bid). There's a 6/27 chance of that happening. If the second–highest bid is 2 or 3, then a reserve bid of 1.5 has no effect on the revenue.

So requiring a reserve bid of 1.5 raises the expected revenue by $(6/27)(0.5) - (1/27) = 2/27$, compared to the expected revenue from an efficient auction. That's not the only way of raising more revenue than an efficient auction : any reserve bid between 1.1666 and 2 will raise revenue, for example. So would a reserve bid between 2.6667 and 3.

$Q10$. What is the sub–game perfect Nash equilibrium to the following game?

There are two players in the game. Firm 1 is a prospective entrant, and firm 2 is an incumbent firm, which already has stores in two markets.

Firm 1 moves first, choosing whether to enter market A, or not to enter.

Firm 2 observes firm 1's first move. If firm 1 chose not to enter, firm 2 has no move to make. But if firm 1 chose (in the initial stage) to enter, then firm 2 chooses whether to accommodate entry or to start a price war in market A.

Firm 1 then chooses whether to enter market B. Firm 1 makes this choice immediately after choosing not to enter market A (if it chose not to enter in the first stage), or makes this choice after observing firm 2's move (if it had chosen to enter in the first stage).

If firm 1 chooses not to enter market B, the game ends. But if firm 1 chooses to enter market B, then firm 2 has a second move, whether to accommodate entry in market B, or to start a price war there. Then the game ends.

The firms' payoffs are the sum of their profits in the two markets.

Firm 1 gets profits of 0 in a market it does not enter, profits of 5 in a market which it entered and in which firm 2 accommodated its entry, and -2 in a market which it entered and in which firm 2 started a price war.

Firm 2 gets profits of 10 in a market in which firm 1 did not enter, profits of 5 in a market in which firm 1 entered and in which it (firm 2) accommodated entry, and profits of -2 in a market in which firm 1 entered and in which it (firm 2) started a price war.

*A*10. The accompanying figure depicts the extensive form of this game.

The subgame perfect Nash equilibrium can be found by backwards induction (starting from the end of the game).

First note that if firm 1 chose to enter market B, then firm 2 will choose to accommodate with his last move. The three nodes in the figure at which firm 2 gets to respond to entry by firm 1 into market B give payoffs of 15, 10 or 3 from accommodation, and 8, 3 and -4 (respectively) from starting a price war, so that accommodation is always better. (That is, better to get \$5 in market B than –\$2.)

Moving up the tree, it then turns out that entry into market B is always best for firm 1, in the three nodes at which she makes this choice. Given firm 1's subsequent best replies, her payoffs are 5, 10 and 3 from entry, and 0, 5 and -2 (respectively) from not entering market B. So she should enter market B, whatever has happened earlier in the game.

Now firm 2's choice of whether to accommodate or not (if firm 1 has entered market A) i to accommodate. Given the subsequent best replies, firm 2 gets a payoff of 10 if it accommodates entry in market A [if firm 1 chose to enter it initially], and 3 if it starts a price war.

Finally, at the initial decision node, firm 1 will get a payoff of 5 if it chooses not to enter, and 10 if it enters, given the subsequent actions by both firms, so it should choose to enter market A.

Therefore, the subgame perfect Nash equilibrium strategies are : firm 1 should enter each market whenever it gets the choice, and firm 2 should accommodate entry in every market, whenever it gets the choice.

Here a price war in market A is not a credible threat, since firm 1 can see that entry into market B will be accommodated, no matter what has happened in market A. Since market A's outcome has no impact on what will happen in market B, any threat by firm 2 to start a price war in market A is not credible.