Production Functions

$$f : \Re^n_+ \rightarrow \Re_+$$

$f(x_1, x_2, \ldots, x_n)$: quantity of output produced from vector of quantities of inputs ; x_1 units of input #1, x_2 units of input #2, etc.

$\frac{\partial f}{\partial x_i}$: marginal product of input i

assumptions : $f(x_1, x_2, \ldots, x_n)$ is strictly monotonic, continuous, quasi–concave

definition : $MRTS$ (marginal rate of technical substitution) : $MRTS_{ij} = \frac{\partial f}{\partial x_i} / \frac{\partial f}{\partial x_j}$

definition : separability : $f(\mathbf{x})$ is separable if $MRTS_{ij}$ does not vary with x_k (where $k \neq i, j$)

more formally : $f(\mathbf{x})$ is **weakly separable** if the n inputs can be divided into S different groups, $N_1, N_2, \ldots, N_S$, and the $MRTS$ between any two inputs in group s does not vary with the quantity of any input in some other group t

example of separability : CES production function

$$f(\mathbf{x}) \equiv (a_1 x_1^\rho + a_2 x_2^\rho + \cdots + a_n x_n^\rho)^{\mu/\rho}$$

where $a_i > 0$, $-\infty < \rho \leq 1$, $\mu > 0$

$$\frac{\partial f}{\partial x_i} = a_i A x_i^{\rho-1} \tag{1}$$

where

$$A \equiv \mu(a_1 x_1^\rho + a_2 x_2^\rho + \cdots + a_n x_n^\rho)^{\mu/\rho - 1}$$

so that

$$MRTS_{ij} = \frac{a_i}{a_j}(\frac{x_j}{x_i})^{1-\rho} \qquad (2)$$

meaning **strong separability**, since $MRTS_{ij}$ doesn't depend on anything but x_i and x_j

Elasticity of Substitution

motivation : price–taking firms will choose input combinations so that

$$\frac{f_i}{f_j} = \frac{w_i}{w_j}$$

where w_i is the unit price of input i

elasticity of substitution σ measures the percentage by which $\frac{x_i}{x_j}$ falls, if $\frac{w_i}{w_j}$ increases by 1%

definition

$$\sigma = -\frac{d(x_i/x_j)}{x_i/x_j} / \frac{d(f_i/f_j)}{f_i/f_j} \tag{3}$$

for CES production function

$$\frac{f_i}{f_j} = (\frac{a_i}{a_j})(\frac{x_i}{x_j})^{\rho-1}$$

so that

$$\frac{d(x_i/x_j)}{d(f_i/f_j)} = \frac{1}{\rho - 1}[\frac{a_j}{a_i}]^{1/(\rho-1)}[\frac{f_i}{f_j}]^{1/(\rho-1)-1}$$

which equals

$$\frac{1}{\rho - 1}[\frac{f_i}{f_j}]^{-1}\frac{x_i}{x_j}$$

so that the elasticity of substitution is $1/(1-\rho)$.