Cost Minimization

Given an output level y, what is the minimum cost of producing it?

$$\text{minimize} \quad \mathbf{w} \cdot \mathbf{x} \qquad \text{subject to} \quad f(\mathbf{x}) \geq y \tag{1}$$

the cost function $C(\mathbf{w}, y)$ is the cost of the input bundle $\mathbf{x}$ which solves minimization problem (1)

The levels $\mathbf{x}$ of the quantities of the inputs which solve problem (1) are called the firm's **conditional input demands**, functions of the vector $\mathbf{w}$ of input prices, as well as on the level y of output required.

seems familiar?

this problem is **exactly** the cost minimization problem which underlies the consumer's expenditure function

with new terminology

required utility level $u \rightarrow$ required level of output y

utility function $u(\mathbf{x}) \rightarrow$ production function $f(\mathbf{x})$

commodity price vector $\mathbf{p} \rightarrow$ input price vector $\mathbf{w}$

commodity vector $\mathbf{x} \rightarrow$ input vector $\mathbf{x}$

$e(\mathbf{p}, u) \rightarrow C(\mathbf{w}, y)$

$\mathbf{x}^h(\mathbf{p}, u) \rightarrow \mathbf{x}(\mathbf{w}, y)$

first–order condition

$$\frac{f_i(\mathbf{x})}{f_j(\mathbf{x})} = \frac{w_i}{w_j} \qquad \text{all} \quad 1 \leq i, j \leq n \tag{2}$$

properties of the cost function

3.2.3 : $C(\mathbf{w}, y)$ is increasing in the output level y (if $\mathbf{w} >> 0$)

3.2.4 : $C(\mathbf{w}, y)$ is increasing in each input price w_i.

3.2.5 : $C(\mathbf{w}, y)$ is homogeneous of degree 1 in $\mathbf{w}$

3.2.6 : $C(\mathbf{w}, y)$ is concave in $\mathbf{w}$

3.2.7 : Shephard's lemma : $\frac{\partial C(\mathbf{w},y)}{\partial w_i} = x_i(\mathbf{w}, y)$

3.3.1 : $\mathbf{x}(\mathbf{w}, y)$ is homogeneous of degree 0 in $\mathbf{w}$

3.3.1 : the n–by–n substitution matrix σ, with entries $\partial x_i(\mathbf{w}, y)/\partial w_j$, is symmetric and negative semi–definite

implication

the conditional demand for any input cannot increase with the price of that input [no ”Giffen inputs”]

homotheticity

$f(\mathbf{x})$ is homothetic if and only if it can be written as $f(\mathbf{x}) = \Phi(g(\mathbf{x}))$

where the function $g : \Re^n_+ \rightarrow \Re$ is homogeneous of degree 1, and $\Phi(\cdot)$ is any transformation mapping $\Re \rightarrow \Re$

homotheticity is a generalization of homogeneity : any function which is homogeneous of degree μ ($0 < \mu < \infty$) is homothetic...but not vice versa

$$f(\mathbf{x}) \equiv \sum_{i=1}^{n} a_i \ln x_i \qquad (3)$$

(where the a_i's are positive constants) is homothetic, but is not homogeneous

homothetic means that the isoquants all have the same slope along any ray through the origin

cost functions for homothetic technologies

unit cost function : $C(\mathbf{w}, 1)$: cost of producing 1 unit of output

with a homothetic production function

$$C(\mathbf{w}, y) = h(y)C(\mathbf{w}, 1) \tag{4}$$

for some increasing function $h(y)$

and

$$\mathbf{x}(\mathbf{w}, y) = h(y)\mathbf{x}(\mathbf{w}, 1) \tag{5}$$

if the production function were homogeneous of degree μ, then $h(y) = y^{1/\mu}$

short run and long run

$C(\mathbf{w}, y)$: **long–run** (total) cost function

short run : fix some input levels

$$SC(\mathbf{w}, \bar{\mathbf{w}}, y, \bar{\mathbf{x}}) = \min_{\mathbf{x}} \mathbf{w} \cdot \mathbf{x} + \bar{\mathbf{w}} \cdot \bar{\mathbf{x}} \quad \text{subject} \quad \text{to} \quad f(\mathbf{x}, \bar{\mathbf{x}}) \geq y \tag{6}$$

$\bar{\mathbf{x}}$: vector of quantities of **fixed** inputs [exogenous]

$\bar{\mathbf{w}}$: vector of unit prices of fixed inputs [exogenous]

$\mathbf{w}$: vector of unit prices of variable inputs [exogenous]

$\mathbf{x}(\mathbf{w}, \bar{\mathbf{x}}, y)$: conditional input demands [endogenous]

$$C(\mathbf{w}, \bar{\mathbf{w}}, y) \leq SC(\mathbf{w}, \bar{\mathbf{w}}, y, \bar{\mathbf{x}}) \tag{7}$$

if $\bar{x_i}$ is long–run cost minimizing for $\bar{\mathbf{w}}, \mathbf{w}, y$ (for all fixed inputs i), then

$$C(\mathbf{w}, \bar{\mathbf{w}}, y) = SC(\mathbf{w}, \bar{\mathbf{w}}, y, \bar{\mathbf{x}}) \tag{8}$$

from (7) and (8), the short–run (total) cost curve must be tangent to the long–run (total) cost curve, at the output level for which the fixed input levels happen to be optimal

"envelope relation"

$$\frac{\partial C}{\partial y} = \frac{\partial SC}{\partial y} + \sum_i \frac{\partial SC}{\partial \bar{x}_i} \frac{\partial \bar{x}_i}{\partial y} \tag{9}$$

$\frac{\partial SC}{\partial \bar{x}_i} = 0$ if fixed inputs are cost–minimizing, so that

$$\frac{\partial C}{\partial y} = \frac{\partial SC}{\partial y} \tag{10}$$

if fixed inputs are cost–minimizing

i.e. $SRMC = LRMC$