Profit Maximization in Perfect Competition

(which will only work if the technology exhibits decreasing returns to scale)

firm's problem : to maximize

$$pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} \tag{1}$$

with respect to its input quantities $\mathbf{x}$

first–order conditions :

$$p\frac{\partial f}{\partial x_i} = w_i \quad i = 1, 2, \cdots n \tag{2}$$

using the cost function

choose an output level y to maximize

$$py - C(\mathbf{w}, y) \tag{3}$$

first–order conditions

$$p = \frac{\partial C}{\partial y} \tag{4}$$

value of maximized profit : $\pi(p, \mathbf{w})$

second–order conditions?

$$\frac{\partial^2 C}{\partial y^2} > 0 \tag{5}$$

why decreasing returns are needed

an example

if $f(\mathbf{x})$ is homogeneous of degree μ

then

$$C(\mathbf{w}, y) = y^{1/\mu} C(\mathbf{w}, 1) \tag{6}$$

so that

$$\frac{\partial C}{\partial y} = \frac{1}{\mu} y^{1/\mu - 1} C(\mathbf{w}, 1) \tag{7}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{1 - \mu}{\mu} \frac{1}{\mu} y^{1/\mu - 2} C(\mathbf{w}, 1) \tag{8}$$

which is positive only if $\mu < 1$

properties of the profit function $\pi(p, \mathbf{w})$

$$\frac{\partial \pi}{\partial p} > 0 \tag{9}$$

$$\frac{\partial \pi}{\partial w_i} \leq 0 \quad i = 1, 2, \ldots, n \tag{10}$$

$\pi(p, \mathbf{w})$ is homogeneous of degree 1 in $(p, \mathbf{w})$

$\pi(p, \mathbf{w})$ is convex in $(p, \mathbf{w})$

Hotelling's Lemma : part 1

$$\frac{\partial \pi}{\partial p} = y(p, \mathbf{w}) \tag{11}$$

what is $y(p, \mathbf{w})$?

the competitive firm's **supply function** : the level of output it will produce, when the output price is p, and when the input prices are $\mathbf{w}$

Proof :

$$\frac{\partial \pi}{\partial p} = \frac{\partial}{\partial p}[py - C(\mathbf{w}, y] = y + (p - \frac{\partial C}{\partial y})\frac{\partial y}{\partial p} \qquad (12)$$

Hotelling's Lemma : part 2

$$\frac{\partial \pi}{\partial w_i} = -x_i^u(p, \mathbf{w}) \qquad (13)$$

where $x_i^u(p, \mathbf{w})$ is the **unconditional** demand for input i

supply curves, unconditional input demand curves

Theorem 3.7 : $\pi(p, \mathbf{w})$ is convex in $(p, \mathbf{w})$

so matrix H of second derivatives of $\pi(p, \mathbf{w})$ is positive definite

$$H = \begin{pmatrix} \frac{\partial^2 \pi}{\partial p^2} & \frac{\partial^2 \pi}{\partial p \partial w_1} & . & . & \frac{\partial^2 \pi}{\partial p \partial w_n} \\ \frac{\partial^2 \pi}{\partial w_1 \partial p} & \frac{\partial^2 \pi}{\partial {w_1}^2} & . & . & \frac{\partial^2 \pi}{\partial w_1 \partial w_n} \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial^2 \pi}{\partial w_n \partial p} & \frac{\partial^2 \pi}{\partial w_n \partial w_1} & . & . & \frac{\partial^2 \pi}{\partial {w_n}^2} \end{pmatrix} \tag{14}$$

Hotelling (1) $\rightarrow \frac{\partial^2 \pi}{\partial p^2} = \frac{\partial y(p, \mathbf{w})}{\partial p}$

so that $\frac{\partial y(p, \mathbf{w})}{\partial p} \geq 0$: supply curves cannot slope down

Hotelling (2) $\rightarrow \frac{\partial^2 \pi}{\partial {w_i}^2} = -\frac{\partial x_i^u(p, \mathbf{w})}{\partial w_i}$

so that $\frac{\partial x_i^u(p, \mathbf{w})}{\partial w_i} \leq 0$: (unconditional) input demand curves cannot slope up