The Lure of the Fundamental Probability Set of Equally Likely Events

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F.N. David's book *Games, Gods and Gambling*[1] ties the origins of probability theory to the formulation of the concept of the fundamental probability set, i.e., a comprehensive set of equally probable outcomes that exhausts the universe of all possible outcomes. Gambling, she noted, has a very Along history, with astragali being found in some of the earliest burial sites of prehistoric peoples and described in some of the earliest written documents of humankind. The term *astragali* as used refers to a variety of knucklebones, hucklebones, and other animal bone configurations that can be used as we would use dice. But being natural objects, not evenly balanced, nor symmetrically configured, they would vary considerably on the frequency of each side landing uppermost. Because the likelihood of particular outcomes varied with each way astragali could land and also varied from astragalus to astragalus, no one, it seems, attempted to construct a general theory of probabilities as they applied to the use of these objects. That is not to say that the objects were not used, and used with great frequency, not just in games of chance, but also as a means of reaching a decision on matters of choice, sometimes of tremendous significance. Since these were used as means of communicating with the gods, it could also be viewed as impious to look further into the odds of particular outcomes.

The traditional marker for the true origin of the theory of probability as a mathematical subject is the famous correspondence between Pascal and Fermat over the proper fair division of the winnings of a game which has to be interrupted before it reaches its designated end result. But just as there is general consensus that this correspondence is the traditional beginning of probability theory there is also a general consensus now that there were a good many precursors and foreshadowings of the concerns raised by Pascal and Fermat. Among the precursors frequently mentioned by those writing about the history of probability is Girolamo Cardano.

Cardano, who lived from 1501 until 1576, is certainly one of the most colorful and interesting characters in all of the history of mathematics. He was also one of the most prolific authors of the entire Renaissance. He wrote well over a hundred books on a variety of topics. Many of them were fairly well lifted right out of other people's works. This kind of plagiarism was not uncommon in Cardano's time—basically the century after Gutenberg—nor was it frowned upon. And I suspect that Cardano was probably a more lively and interesting author than those he borrowed from. Some of his works were groundbreaking and important for mathematics in Europe; in particular his *Ars Magna* was a significant step in the advance of algebra.

Many of his works remained unpublished at his death, including his most important writing on games of chance, the *Liber de Ludo Aleae*, which was not finally published until a collection of his works was published in 1663, almost a century after his death. It is in this work that Cardano indicates how one can build up a calculation of the chances of a specified outcome by counting up all the combinations of favourable results. The tacit assumption here is that all of these basic results are equally probable. In the particular case illustrated, namely the cast of a die, if you assume that the dice are well made, the argument from symmetry alone suffices as *a priori* justification for the assumption that any one result is as likely as any other. That said, one can then go on to build up the probability of a more complex result from the "atomic" individual results. To quote from Cardano's text:

> One-half the total number of faces always represents equality; thus the chances are equal that a given point will turn up in three throws, for the total circuit is completed in six, or again that one of three given points will turn up in one throw. For example I can as easily throw one, three or five as two, four or six.[2]

According to Dr. David, this is, as far as she knows, the first time that "the abstraction from empiricism to theoretical concept is made."[3] No doubt, Cardano saw pretty quickly that if you can reduce all possible outcomes of a game of dice, or any other game, to a collection of equally likely component outcomes, then it becomes a straightforward matter of tallying up all the favourable outcomes and comparing them to the full set of all outcomes to find the theoretical probability of the outcome of interest. In fact it is so much more straightforward to make this sort of calculation that it really pays to take the time to reduce any gaming situation to a set of equiprobable outcomes, the fundamental probability set.

The assertion I want to make is that throughout the history of probability theory it has been all too inviting to find such a set. It is very tempting to see any situation involving uncertain outcomes as being reducible to a fundamental probability set, because then we can proceed to use all the wonderful mathematical results that have been derived in probability theory to determine the likelihood of the event before us. The lure of the fundamental probability set is high.

In fact, I think even Cardano was lured by its manifest simplicity to use it when it really wasn't justified.

A later section of his *Liber de Ludo Aleae* discusses gambling games that use knucklebones instead of dice. Knucklebones, one of the commonest forms of astragali, were used for centuries as gambling devices in place of dice. The typical knucklebone can land in any of four possible positions. But each knucklebone is distinct from any other. They are all irregular. While they can land in four possible positions, it would be a rare coincidence for any one to do so with equal frequency. Cardano seems to have looked past this possibility. When discussing the equiprobable outcomes with dice, Cardano adds the caveat "if the die be honest," but when he discusses the extension of this kind of analysis to astragali and other gambling devices, he neglects to re-assert the proviso about honesty. Perhaps Cardano himself viewed his probability assertions as interesting in their own right and was perfectly willing to leave them as commentary about an ideal situation. But the suggestion that they apply to situations where the basic outcomes are patently not equally likely is insidious. How tempting it is to contemplate a neat mathematical model and to generalize it to cover situations where it does not quite fit.

But as mentioned previously, the generally agreed upon official beginning of probability theory in anything like the form that we now know it is the famous correspondence between Blaise Pascal and Pierre de Fermat in 1654 over the Problem of Points. The story there is well known: the Chevalier de Méré, an inveterate gambler, approached Blaise Pascal with some questions about a theoretical situation that can arise in games of chance. The situation that troubled de Méré concerned the proper and just division of the stakes in a game of chance that was interrupted before the stipulated conditions had been reached that would determine the winner. The particulars were that it was a game of chance, where the contestants were equally matched; the game proceeded in discrete rounds and a winner was determined when one of the contestants won a specified number of rounds. Supposing that the game was in progress and a number of rounds had already been determined, but not enough to establish a winner when it had to be discontinued. How should the stakes be divided?

Pascal thought the matter over, but was not sure he had found the correct solution. On the advice of a colleague, Pascal wrote a letter to Pierre de Fermat, whom he did not know personally, but who was widely regarded as France's most brilliant mathematician. Fermat replied and several letters were exchanged back and forth about the problem, some of which have survived. This exchange of letters is the traditional event marking the birth of probability theory.

To underscore that assessment, let me quote from Keith Devlin's book *The Unfinished Game*, a book devoted entirely to assessing the importance and context of Pascal's original letter to Fermat on August 24, 1654. Devlin states "When completed, the letter would come to less than three thousand words, but it would change human life forever. It set out, for the first time, a method whereby humans can predict the future."[4] Devlin was, to be sure, justifying his decision to write a whole book about one letter, but in doing so, he at most overstated the general consensus that it was when a couple of the most original mathematical minds turned their attention to a theoretical question about gambling that a whole new outlook on managing uncertainty took root leading to extensive and comprehensive applications of tools for calculating the odds at a gaming table to questions about optimal decisions in all areas of human inquiry. In short, to play along with this caricature, as a consequence of this letter, the artificially constructed environment of gaming became the mathematical model for the universe itself.

The next step in developing a theory of probability based upon counting up equally possible outcomes was taken by the Dutchman Christiaan Huygens in 1656 after he had made a visit to Paris and learned of the Pascal-Fermat correspondence. He wrote a short pamphlet on the probability of gaming, *On Reasoning in Games of Chance,* which took the upshot of the Pascal-Fermat correspondence and formalized it in a treatise that showed how to calculate probabilities in games of chance using combinations and permutations. This work, by the way, is sometimes called the first explicit text on probability. But it is entirely concerned with games of chance. In was published in the following year as an appendix to a work by Frans van Schooten—probably best known now as the person who translated expanded Descartes' *La Geometrie* and made it into a major text in analytic geometry. The Huygens pamphlet was then separately translated into Dutch, English, and French, and published in a variety of formats, including becoming Book I of Jacob Bernoulli's *Ars Conjectandi.*[5]

By the time of Pascal and Fermat, there already was a small body of mathematical theory about dice games, roulette, and some other artificially constructed gambling venues for gaming, and there certainly was an extensive amount of gambling going on in the world, with or without a theoretical underpinning for it. In a very short time probability theory emerged as significant branch of mathematics. And, as has been extensively documented by such people as Ian Hacking and Lorraine Daston, alongside the mathematics of gaming arose another wing of probability concerned with questions of knowledge and judgment, sometimes called *epistemic* probability, with roots in matters of jurisprudence. That side of probability thinking is a rich and complex story that certainly affected the development of the subject as a whole. However, that is not my primary concern here. What I am interested in understanding is how the enumeration of equipossible events became the basic tool for the measurement of probability in all assessments of risk or uncertainty in the world. To put that in metaphorical terms, what I want to know is how the universe came to be seen as a casino, and to question whether it makes a good fit.

Accounts of the widening of mathematical probability from an analysis of games of chance to phenomena of the wider world generally point to the collection of statistical data as the next important event. John Graunt's book *Natural and Political Observations on the Bills of Mortality*[6] appeared in England in 1662. Graunt used actual mortality statistics to make predictive statements about life-expectancy. No doubt, this suggested other ways that that samples of statistical data could be used to fix the parameters of a probability distribution applicable to unknown data that was deemed similar in kind.

This is exactly what occurred to Chistiaan Huygens's younger brother, Lodewijk Huygens. It is said that the Christiaan Huygens had been sent a copy of Graunt's book on mortality statistics by the president of the Royal Society, but did not give it more than a cursory glance[7] until Lodewijk Huygens suggested to Christiaan that one could use Graunt's tables of life-expectancy to calculate what we would now call the present value of life annuities.

Christiaan then took a step which I view as characteristic of what was to come in the history of statistical reasoning, he fitted the messy data of real-world statistics into the Procrustean bed of the probability of gambling. Since the mathematics of probability had been conceived and thought through for games of chance, Huygens decided that the way to analyze a mortuary table was to conceive it as a lottery where the value of each ticket corresponded to entries on the mortuary table.[8]

Huygens did not go far with this, but another Dutchman, Johan De Witt, ran with Huygens' conception in his *Treatise on Life Annuities*[9] in 1671. To quote from Lorraine Daston, "De Witt's originality lay in his attempt to estimate the probability of death as a correlate of age, and in his extension of Huygens's calculus of expectations to a new class of problems."[10]

Here comes the inevitable too-good-to-be-true shortcut that entered statistical reasoning at this early stage, and, unless I am mistaken, has not ceased to infect its results ever since: De Witt had to cope with assigning a probability to outcomes where he did not have data, such as the likelihood of dying at different ages of life. He solved this problem by assigning equal probabilities for the risk of death for everyone from the age of three to fifty-three, and then using proportional probabilities or educated guesswork for ages out of this range.[11]

At this point, from my point of view, the trail goes somewhat cold. I have not found others wrestling with unknown data by just assuming a pervasive equality. Or, I could express this another way: instead of saying the trail goes cold, I could say that the trail becomes paved over with clear signposts. That is, it seems to me that the conjuring trick of equipossibility became very popular very fast and as a result, no longer seemed to need justification.

In order to be able to use all the neat mathematical formulae from the probability of games of chance that are based on the symmetry of the elements of gambling: roulette wheels, dice, coins, playing cards, etc., where the likelihood of any one outcome is the same as the likelihood of any other, you need to have equipossible events in real life, or, you need to know the likelihood of different outcomes *a priori* before you can apply the formulae from the gaming room. Sometimes you do have this data, but in many cases you don't. The solution is easy: where data fails you, make the assumption of equal probability. The particular version of this that troubles me the most is the assumption of a fifty:fifty chance when there is no guiding information at all. This little addition makes it possible to calculate probable outcomes for all sorts of events after the data runs out. What I find extraordinary is how quickly this idea caught on and became buried in the foundations of probability theory.

Laplace embodied the fifty:fifty idea in his very definition of probability:

> The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence.[12]

Among my favorite iterations of this principle is the perfectly serious debate that went on in the eighteenth and nineteenth centuries on calculating the chances of the sun rising the next day. The assertion was that as the sun had risen successively every day since the beginning of human records and since that amounted to roughly 1,825,000 days, then the odds of the sun rising the next day would be 1,825,000 plus 1 divided by 1,825,000 plus 2, i.e. $(m+1)/(m+2)$. The sole philosophical justification of this bizarre reasoning is the infamous *Principle of Indifference* that states that if there is no reason to favour one outcome over another, the chances of all possible outcomes are equal.

This is what I call the "lure" of the fundamental probability set. If you adopt this ever so convenient axiom, it enables you to apply a whole panoply of powerful mathematical theorems derived for symmetric games of chance to the messy world of nature. It amounts to a simple assertion that when you are totally ignorant, suddenly, the casino comes to the rescue, and you have an even chance of whatever you don't know anything about coming out as you might wish it to.

[1] F. N. David, *Games, Gods and Gambling: The Origins and History of Probability and Statistical Ideas from the Earliest Times to the Newtonian Era*, London: Charles Griffin & Co., 1962.

[2] *Ibid.*, p. 58. David does not give the exact page reference for Cardano, but says this is from the “crucial chapter” “On the cast of one die” in Gould’s translation of *Liber de Ludo Aleae.*

[3] *Ibid.*

[4] Keith Devlin, *The Unfinished Game: Pascal, Fermat, and the Seventeenth Century Letter that Made the World Modern,* New York: Basic Books, 2008.

[5] Christiaan Huygens, *De Ratiociniis in Ludo Aleae* [1657] in Huygens, *Oeuvres Complètes,* The Hague: Société Hollandaise des Sciences, 1888-1967.

[6] John Graunt, *Natural and Political Observations Mentioned in a Following Index, and Made Upon the Bills of Mortality,* 3rd ed. London: John Martyn and James Allestry, 1665 (1st ed 1662).

[7] Devlin, p. 101.

[8] *Ibid.,* p. 102.

[9] Johan de Witt, *Waerdye van Lyf-Renten* [1671], in *Die Werke von Jakob Bernoulli,* vol. 3, pp.328-350. English translation by F. Hendriks in Robert G. Barnwell, *A Sketch of the Life and Times of John De Witt,* New York, 1856.

[10] Lorraine Daston, *Classical Probability in the Enlightenment*, Princeton: Princeton University Press, 1988, p. 27.

[11] *Ibid.*

[12] Laplace, *A Philosophical Essay on Probabilities,* [1814] trans. by Frederick Wilson Truscott and Frederick Lincoln Emory, Dover, 1951, p. 6. Cited in Daston, p. 29.