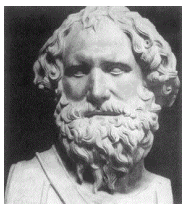
The Twilight of Ancient Greek Mathematics

Hellenistic mathematics post-Euclid and the discovery of mathematical thinking in the Orient

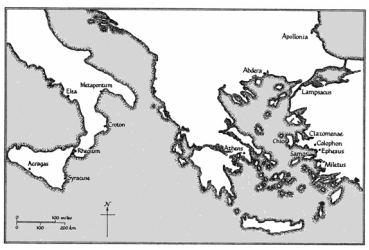
Archimedes, the greatest of all the ancient mathematicians

- Euclid had the greatest influence of all ancient mathematicians (perhaps of all mathematicians of all time), but is not regarded as the most original nor the most creative of ancient mathematicians.
- That honour, by general agreement, is conferred on Archimedes of Syracuse.



Archimedes, ca. 287-212 B.C.E.

- Archimedes was born and lived in the Greek colony of Syracuse, on the southeast coast of Sicily.
- He was the son of an astronomer, Phidias.
- He was a contemporary of Eratosthenes.
- He was a generation or two younger than Euclid.



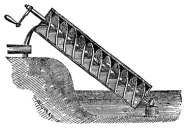
Archimedes

- Syracuse was the largest city in the Hellenistic world.
- Archimedes was related to the king of Syracuse, Hieron II, pictured on the coin at the right.
- It is likely that Archimedes travelled to Egypt and studied at Alexandria. He corresponded with several scholars there, but he spent his most productive years in Syracuse, under the patronage of Hieron.

Archimedes' fame

- Archimedes was famous in ancient times for his mathematical writings, his mechanical inventions, and his ingenious military skills put to use in the defence of Syracuse from Roman attack.
- He is famously said to have died in the siege of Syracuse in 212 BCE when slayed by a Roman soldier who found him immersed in mathematical calculation and ordered him to stop.

Archimedes' exploits



- The Archimedean screw for raising water to a higher level.
- The compound pulley to increase leverage. Archimedes is said to have dragged a heavy ship onto shore by attaching a rope to it, going through a compound pulley, and just pulling the rope himself without any help. This story is attached to his famous quote: "Give me a place to stand and I will move the Earth."

The Crown Problem

- King Hieron was said to have had a crown made for himself.
- He gave a fixed amount of gold to a goldsmith who was to have shaped it into a crown (in the shape of a laurel wreath).
- Hieron was suspicious, however, that the goldsmith may have kept part of the gold for himself and substituted an equal weight of another less valuable metal, such as silver or copper, and alloyed them together.
- Hieron asked Archimedes to figure out if he had been cheated.

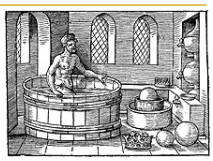
What Archimedes knew:

- Gold is far heavier than either silver or copper.
  - E.g. gold weighs 19.3 g/ml. Copper weighs 8.92 g/ml.
- Volume of the crown:
  - A crown of solid gold would have the same volume (take up the same amount of space) as the original amount of gold.
  - An alloyed crown would have a greater volume.

Measuring the volume of the crown:

- Euclid's mathematics provided the means to measure the volume of an object with a nice regular shape, e.g. rectangular, conical, spherical, etc.
- The crown has an entirely irregular shape.
- The volume of the crown could be measured if it were hammered back into the form of a rectangular brick. – Destroying the crown!

Eureka!



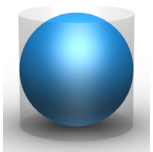
- Archimedes went to the baths.
- There it occurred to him that when he lowers his body into the bath, it pushes out of the way a volume of water equal to the volume of Archimedes' body.
- If the crown were lowered into water, it too would push out a volume of water equal to the volume of the crown.
- The volume of the water can therefore be measured.

Archimedes' "Eureka" – Greek for "I have found it!"

- Archimedes knew:
  - Gold weighs 19.3 g/ml. Copper weighs 8.92 g/ml.
  - Suppose he had weighed the crown and found that it weighed exactly 2 kg, being the weight of gold that the king had given to the goldsmith.
- Upon carefully immersing the crown in a vessel filled to the absolute brim with water he finds it pushes exactly 115.68 ml of water out of the vessel.
  - What can Archimedes conclude about the crown?

Archimedes' *On the Sphere and the Cylinder*

- Archimedes was proudest of his work on the geometry of the sphere and the cylinder, published in two books, totalling 53 propositions.
- Among these are:
  - The surface of a sphere is four times the area of a great circle of the sphere. [i.e. $S=4\pi r^2$]
  - If about a sphere there is circumscribed a cylinder whose height is equal to the diameter of the sphere, then the volume of the cylinder is 3/2 of the volume of the sphere; and the surface of the circumscribing cylinder, including its bases, is 3/2 of the surface of the sphere.
    - Archimedes requested that this diagram appear on his tombstone.

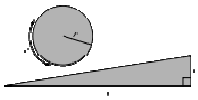


Archimedes' *The Measurement of a Circle*

- This work was Archimedes' best known work during the Middle Ages. It was the first to be translated into Latin. It is short, only three propositions, and probably was originally part of a longer work.
- All three propositions are, in effect, calculations of a value for π.

Archimedes' *The Measurement of a Circle, 2*

- Proposition 1: The area of any circle is equal to the area of a right triangle in where one side extending from the right angle is equal to the radius of the circle and the other side extending from the right angle is equal to the circumference.
  - Note that this amounts to saying that the area of a circle is equal to πr^2.

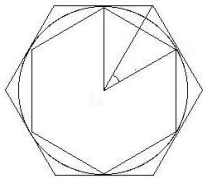


Archimedes' *The Measurement of a Circle, 3*

- Proposition 2: The area of a circle is to the square on its diameter as 11 to 14, very nearly. [*This is clearly out of place, since its proof depends on proposition 3.*]
- Proposition 3: The circumference of any circle exceeds three times its diameter by a part that is less than 1/7 but more than 10/71 of the diameter.
  - Leading to the useful approximation of 22/7 for the value of π.
  - Note: The symbol π for the ratio of the circumference to the diameter of a circle was not used by Archimedes. It was introduced in 1706 by William Jones in his book *New Introduction to the Mathematics*. Later in 1748, Leonhard Euler adopted it and then it entered general usage.
  - Why π? Because it is the first letter of the Greek word *perimetros*, meaning "perimeter."

How did Archimedes calculate π?

- He began by inscribing and circumscribing hexagons inside and outside the circle. The perimeter of the hexagons is easy to calculate.
  - The perimeter of the inner hexagon is 6 times the radius of the circle.
  - The perimeter of the outer hexgon is 6 times $2(\sqrt{3})r$. (Using the Pythagorean theorem.)
  - The circumference of the circle lies clearly between these two perimeters.
- From these hexagons, Archimedes constructed dodecagons (12-sided figures) both inside and outside the circle. These lie even closer to the circumference of the circle. These perimeters can be calculated in a straightforward manner.
- He continued to bisect the sides, making 24-sided figures, then 48-sided, and finally 96 sided figures both inside and outside the circle. Calculating these perimeters he found the inside perimeter to be 3 10/71 of the diameter of the circle and the outer perimeter to be 3 1/7 of the diameter.
  - Hence the circumference lies between these two limits. This gives a value for π of between 3.140845 and 3.142857.



Back in Alexandria…

- The great centre of learning at the Museum in Alexandria fell on hard times after the Romans annexed Egypt.
- Octavian entered Alexandria in 30 BCE, signalling the end of any independent rule of Egypt, but Alexandria remained a prosperous centre for another 200 years.
- By 200 CE, the city was plagued by unruly mobs and street riots.

The "Silver" Age of Greek Mathematics

- The "Golden" Age of Greek Mathematics ended with Euclid and Archimedes. Then there was very little original mathematics for several hundred years.
- Then, two great Greek mathematicians appeared, living under Roman rule:
  - Diophantus
  - Pappus

Diophantus

- Little is known about the life of Diophantus except that he lived in Alexandria around the year 250 CE.
- He wrote in Greek and embraced the Greek style of theoretical mathematics, but he was likely a Hellenized Babylonian.
- Typically all we know of his personal career comes from a mathematical riddle.

The Life of Diophantus

- His boyhood lasted for 1/6 of his life.
- He grew a beard after 1/12 more.
- He married after 1/7 more.
- His son was born 5 years later.
- The son lived to half his father's age.
- Then the father died four years later.
- How long did Diophantus live?

Diophantus', *Arithmetica*

- Despite the name, this work by Diophantus is the earliest known treatise devoted to algebra.
- Like Euclid's *Elements* it was written in 13 volumes. Unfortunately, only 6 have been preserved, the others having been lost at an early date.
- There are other works attributed to Diophantus, but they have not survived.

The *Arithmetica*

- What we have of the work is an assortment of 189 problems with solutions.
- A "solution" for Diophantus is a rational, positive number, expressible as a ratio, e.g. 3, 27, 13/259.
- Problems involve powers up to 6.
- Notation was nothing like ours.

The *Arithmetica*, 2

- Diophantus introduced symbolic notation into algebra. Before him any algebraic problem was expressed entirely in words (like the riddle about his life).
- Numbers were expressed in the Greek decimal system, but the notation used letters instead of numerals.
- Diophantus did not recognize negative numbers, hence the equation 4x+20=4 he describes as "absurd" because the solution would be - 4.

Pappus of Alexandria, circa 290 – 350 C.E.

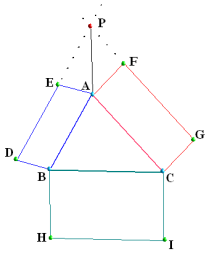
- Little is known of the life of Pappus, except that he lived approximately at the time shown above and he taught at the Museum at Alexandria.
- His main work, the *Mathematical Collection*, is a compendium on much of the work of his predecessors at the Museum and elsewhere in Greek civilization. Often this is the only source for this work, so without the *Mathematical Collection*, all this would be lost.
- He also wrote an extensive commentary on Euclid's *Elements* and another on Ptolemy's *Almagest*. Of these only the commentary on the *Almagest* has survived, though the commentary on Euclid has been widely quoted in a later commentary by Proclus (d. 485 C.E.).

Pappus' *Mathematical Collection*

- The *Collection* was written in eight books, but Book I and part of Book II are missing.
- Pappus also reported extensively on ancient attempts to solve the three classical conundrum problems of antiquity:
  - Quadrature of the circle, duplication of the cube, and the trisection of an angle.
  - He produced all the ancient "solutions" to these problems and then concluded that all of these problems were insoluble in the terms formalized—a result only proven finally in the 19th century.

Pappus' *Mathematical Collection*, 2

- In addition to reporting on the work of his precedessors, the *Collection* contains Pappus' own generalization of the Pythagorean theorem to any triangle, known as the Pappus Area Theorem:
  - Take a triangle ABC, generate on the two sides two parallelograms DEAB and AFGC. Extend ED and FG. They intersect at the point P.
  - Construct a parallelogram BHIC using the triangle side BC and BH = CI = PA, where BH and CI are parallel to the line PA.
  - Then for the areas of the parallelograms DEAB, AFGC, BHIC we have: DEAB + AFGC = BHIC



What about China?

- It seems impossible that the great civilization in China did not also make great advances in mathematics over the vast span of time from its founding to, say, the middle ages.
- Unfortunately, there are very few surviving documents that show what their mathematics was like.

The main exception

- One mathematical work survived from antiquity from China.
- Known as the *Nine Chapters on the Mathematical Art*.
  - It is the oldest textbook in existence on arithmetic.
  - Its date and origin is unknown, but it appears to be from about the time of Euclid (300 BCE).

The *Nine Chapters*

- An earlier Chinese mathematical work, *Arithmetic Classic of the Gnomon and the Circular Paths of Heaven*, which is about astronomy and cosmology is very mystical in character.
- The *Nine Chapters*, on the other hand, is a practical work on how to solve certain kinds of everyday problems.
  - Chapter titles: “Field Measurement,” “Distribution by Proportion,” “Fair Taxes,” etc.

The *Nine Chapters*, 2

- Like Euclid’s *Elements*, the *Nine Chapters* appears to gather together much of what was known about mathematics in the civilization.
- It appears to be a compilation by many different authors.
- All the original copies were destroyed in the famous Burning of the Books in 213 BCE.
- The present existing text was pieced together from surviving fragments and given a commentary by the mathematician Liu Hui in 263 CE.

The *Nine Chapters*, 3

- Like Euclid's *Elements*, the *Nine Chapters* was used widely as a textbook.
- Chinese universities decreed it to be the official mathematics textbook for civil service examination preparation in the 7th century.
- It was one of the first printed books, using the wood block technique, in 1084.

The *Nine Chapters*, 4

- Unlike Euclid's work, the *Nine Chapters* gives neither geometrical analysis nor logical proofs, but instead provides procedures for solving various sorts of problems, by way of example.
  - This was typical of almost all early mathematical works except Euclid.
- An interesting example is the procedure for solving simultaneous linear equations.

Simultaneous equations in the *Nine Chapters*

- **The following is one of the problems in the 8th chapter:**
- "There are three grades of corn. After threshing, 3 bundles of top grade, 2 bundles of medium grade, and 1 bundle of low grade will make 39 dou [a measure of volume]. Two bundles of top grade, 3 bundles of medium grade, and 1 bundle of low grade will produce 34 dou. The yield of 1 bundle of top grade, 2 bundles of medium grade, and 3 bundles of low grade is 26 dou."
- "How many dou are contained in each bundle of each grade?"

Solution

- Express all the mixture formulae as an array.
- Then add, subtract, and/or multiply the columns to get a simplification, which can be solved by simple calculation and substitution.

Top	1	2	3
Med	2	3	2
Low	3	1	1
Dou	26	34	39

Liu Hui's other work

- The *Nine Chapters* is known to later times because Liu Hui pieced together the surviving fragments and wrote a commentary to it.
- Liu Hui also wrote another work called the *Sea Island Mathematical Manual*, which is a treatise on surveying containing only nine problems. It appears to have been intended as a supplement to the *Nine Chapters*, but it was separated from it and treated as a separate text by the 7th century.

The Sea Island Mathematical Manual

- As the name suggests, *Sea Island* discusses how to determine distances to inaccessible points using geometric relationships, rather like Thales' determination of the distance from the shore of a ship at sea.
- The problems generally require two, or sometimes up to four observations.
  - The principles involved are geometric, but the solutions involve what amounts to algebra.

The title problem

- The work takes its name from the original problem in the book, which is given here to illustrate the method:
  - "There is a sea island that is to be measured. Two poles that are each 30 feet high are erected on the same level, 1000 paces [1 pace=6 feet] apart, so that the rear pole is in a straight line with the island and the first pole. If a man walks 123 paces back from the first pole, the highest point of the island is just visible through the top of the pole when he views it from ground level. Should he move 127 paces back from the rear pole, the summit of the island is just visible through the top of the pole when seen from a point on ground level. It is required to find the height of the island and its distance from the nearer pole."

And India?

- As with China, much early work has been totally lost, leaving only speculation as to what early Indian mathematical thinking was like.
- What we know of Indian mathematics is due to the lines of communication established between India and the rest of the world as a result of Alexander the Great's invasion in the late 4th century B.C.E.

Greek, and then Chinese influence in India

- The earliest works which have come down to us show the direct influence of Greek mathematics in the beginning, and then later, an absorption of some Chinese ideas.
- Nevertheless, the Indians developed an independent and, in all topics except geometry, superior to what was being done elsewhere.

The problem of symbols

- Much of Indian mathematics concerned what we would now call number theory and algebra, but they did not develop an algebraic symbolism that would have permitted an economic and unambiguous way of expressing their ideas.
- Instead, mathematical problems were expressed in verse, using metaphorical and ornate language. Little attention was given to mathematical proof or logical demonstration.
  - Frequently the only aid to understanding would be a diagram with the comment: "Behold!"

Some Hindu mathematicians: Arabhata (born 476 C.E.)

- Arabhata was primarily an astronomer. Among his works is an interesting and remarkably accurate formula for the calculation of the ratio between the circumference and the diameter of a circle, which we call π.
  - Add four to one hundred, multiply by eight and then add sixty-two thousand; the result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given.

Some Hindu mathematicians: Brahmagupta (circa 598-670 C.E.)

- Another Indian astronomer, Brahmagupta, also wrote on pure mathematics topics.
- He is the likely source for the introduction of the concept of *zero* as a number. In his main work, the *Brahmasphutasiddhanta*, Brahmagupta discusses the properties of this new number, zero:
  - *When zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero.*
  - *A debt minus zero is a debt.*
    A fortune minus zero is a fortune.
    Zero minus zero is a zero.
    A debt subtracted from zero is a fortune.
    A fortune subtracted from zero is a debt.
    The product of zero multiplied by a debt or fortune is zero.
    The product of zero multiplied by zero is zero.
    The product or quotient of two fortunes is one fortune.
    The product or quotient of two debts is one fortune.
    The product or quotient of a debt and a fortune is a debt.
    The product or quotient of a fortune and a debt is a debt.

Some Hindu mathematicians: Bhaskara (1114-1185)

- The leading Indian mathematician of the 12th century. Known both as a mathematician and an astrologer.
- By this time Indian mathematics was known in China and vice-versa, and there are strong similarities between the two.
- Much of Indian mathematics had evolved into mind teasers, posed just for the pleasure of working them out, rather than for some utilitarian purpose. Typical problems involved elaborate mathematical puzzles with no special purpose.
- Bhaskara's main work *Siddhanta Siromani*, was translated into Arabic in 1587 and became known in the West that way. Part I of that work is named after and dedicated to his daughter, Lilavati. Many of the problems posed in it are addressed to her, e.g.:
  - One-fifth of a swarm of bees is resting on a kadaba bush and a third on a silindha bush; one-third of the difference between these two numbers is on a kutaja, and a single bee has flown off in the breeze, drawn by the odor of a jasmine and a pandam. Tell me beautiful maiden, how many bees are there?

More on Brahmagupta

- In addition to the concept of zero, Brahmagupta described an algorithm for multiplying large numbers together which is very much like the system we teach today, though the notation system is different.
- Brahmagupta used the Hindu system of numerals that later was adopted by the Arabs and which we call Arabic numerals.
- In a backward step, Brahmagupta used $\sqrt{10}$ as a "neat value" for π, which, being approximately 3.1623, is not nearly as good as Aryabhata's value.
- Brahmagupta provided solutions to a great many algebraic and geometric problems, especially those of relevance to astronomy.
- He calculated the length of the solar year at 365 days, 6 hours, 5 minutes, and 19 seconds, which is remarkably close to today's figure of 365 days, 5 hours, 48 minutes, and 45 seconds.