Pythagoras

Numbers as the ultimate reality

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1

Pythagoras of Samos

- Born between 580 and 569. Died about 500 BCE.
- Lived in Samos, an island off the coast of Ionia.

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2

Pythagoras and the Pythagoreans

- Pythagoras himself lived earlier than many of the other Pre-Socratics and had some influence on them:
  - E.g., Heraclitos, Parmenides, and Zeno
- Very little is known about what Pythagoras himself taught, but he founded a cult that promoted and extended his views. Most of what we know is from his followers.

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3

The Pythagorean Cult

- The followers of Pythagoras were a close-knit group like a religious cult.
- Vows of poverty.
- Secrecy.
- Special dress, went barefoot.
- Strict diet:
  - Vegetarian
  - Ate no beans.

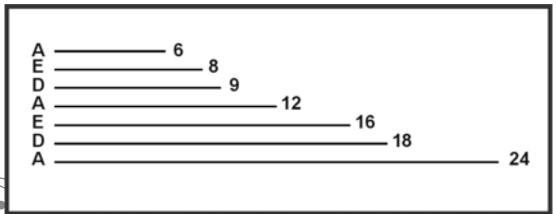
Everything is Number

- The Pythagoreans viewed number as the underlying structure of everything in the universe.
  - Compare to Thales' view of water, Anaximander's *apeiron*, Anaximenes' air, Heraclitos, change.
- Pythagorean numbers take up space.
  - Like little hard spheres.

Numbers and Music

- One of the discoveries attributed to Pythagoras himself.
- Musical scale:
  - 1:2 = octave
  - 2:3 = perfect fifth
  - 3:4 = perfect fourth

Numbers and Music, contd.

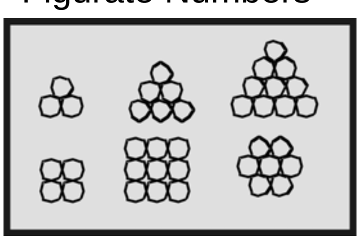


- Relative string lengths for notes of the scale from lowest note (bottom) to highest.
- The octave higher is half the length of the former. The fourth is ¾, the fifth is 2/3.

Geometric Harmony

- The numbers 12, 8, 6 represent the lengths of a ground note, the fifth above, and the octave above the ground note.
  - Hence these numbers form a “harmonic progression.”
- A cube has 12 edges, 8 corners, and 6 faces.
  - Fantastic! A cube is in “geometric harmony.”

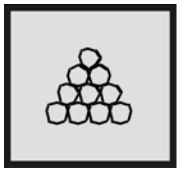
Figurate Numbers



- Numbers that can be arranged to form a regular figure (triangle, square, hexagon, etc.) are called figurate numbers.

The Tetractys

- Special significance was given to the number 10, which can be arranged as a triangle with 4 on each side.
- Called the tetrad or *tetractys.*



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10

The significance of the Tetractys

- The number 10, the *tetractys*, was considered sacred.
- It was more than just the base of the number system and the number of fingers.
- The Pythagorean oath:
  - "By him that gave to our generation the Tetractys, which contains the fount and root of eternal nature."

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11

Pythagorean Cosmology

- Unlike almost every other ancient thinker, the Pythagoreans did not place the Earth at the centre of the universe.
- The Earth was too imperfect for such a noble position.
- Instead the centre was the "Central Fire" or, the watchtower of Zeus.

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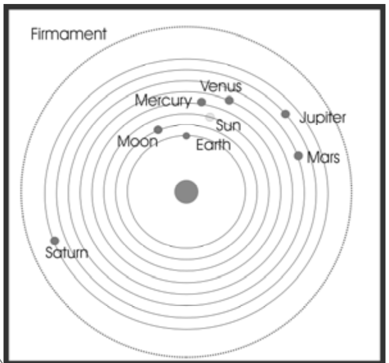
12

The Pythagorean cosmos -- with 9 heavenly bodies



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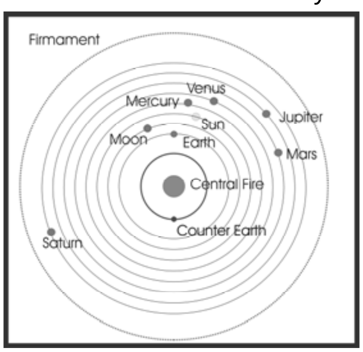
13

The Pythagorean Cosmos and the Tetractys

- To match the tetractys, another heavenly body was needed.
- Hence, the counter earth, or *antichthon*, always on the other side of the central fire, and invisible to human eyes.



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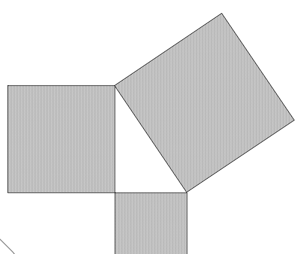
14

The Pythagorean Theorem



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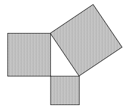
15

The Pythagorean Theorem, contd.

- Legend has it that Pythagoras himself discovered the truth of the theorem that bears his name:
  - That if squares are built upon the sides of *any* right triangle, the sum of the areas of the two smaller squares is equal to the area of the largest square.



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16

Well-known Special Cases

- Records from both Egypt and Babylonia as well as oriental civilizations show that special cases of the theorem were well known and used in surveying and building.
- The best known special cases are
  - The 3-4-5 triangle: $3^2+4^2=5^2$ or 9+16=25
  - The 5-12-13 triangle: $5^2+12^2=13^2$ or 25+144=169

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17

Commensurability

- Essential to the Pythagorean view that everything is ultimately number is the notion that the same scale of measurement can be used for everything.
- E.g., for length, the same ruler, perhaps divided into smaller and smaller units, will ultimately measure every possible length exactly.
- This is called *commensurability*.

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18

Commensurable Numbers

- Numbers, for the Pythagoreans, mean the natural, counting numbers.
- All natural numbers are commensurable because the can all be “measured” by the same unit, namely 1.
  - The number 25 is measured by 1 laid off 25 times.
  - The number 36 is measured by 1 laid off 36 times.

Commensurable Magnitudes

- A magnitude is a measurable quantity, for example, length.
- Two magnitudes are commensurable if a common unit can be laid off to measure each one exactly.
  - E.g., two lengths of 36.2 cm and 171.3 cm are commensurable because each is an exact multiple of the unit of measure 0.1 cm.
    - 36.2 cm is exactly 362 units and 171.3 cm is exactly 1713 units.

Commensurability is essential for the Pythagorean view.

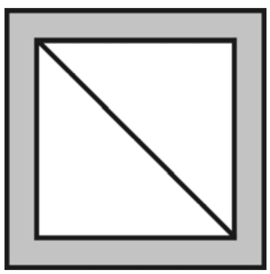
- If everything that exists in the world ultimately has a numerical structure, and numbers mean some tiny spherical balls that occupy space, then everything in the world is ultimately commensurable with everything else.
- It may be difficult to find the common measure, but it just *must* exist.

Incommensurability

- The (inconceivable) opposite to commensurability is *incommensurability,* the situation where no common measure between two quantities exists.
- To prove that two quantities are commensurable, one need only find a single common measure.
- To prove that quantities are *in*commensurable, it would be necessary to prove that no common measures could possibly exist.

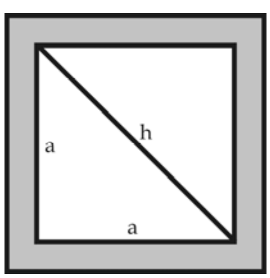
The Diagonal of the Square

- The downfall of the Pythagorean world view came out of their greatest triumph the Pythagorean theorem.
- Consider the simplest case, the right triangles formed by the diagonal of a square.



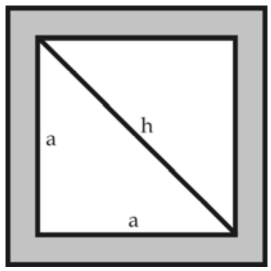
Proving Incommensurability

- If the diagonal and the side of the square are commensurable, then they can each be measured by some common unit.
- Suppose we choose the largest common unit of length that goes exactly into both.



Proving Incommensurability, 2

- Call the number of times the measuring unit fits on the diagonal *h* and the number of time it fits on the side of the square *a*.
- It cannot be that *a* and *h* are both even numbers, because if they were, a larger unit (twice the size) would have fit exactly into both the diagonal and the side.



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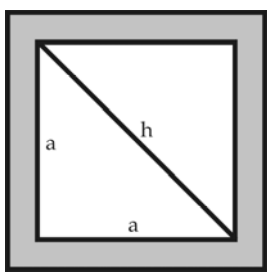
25

Proving Incommensurability, 3

- By the Pythagorean theorem, $a^2 + a^2 = h^2$
- If $2a^2 = h^2$ then h^2 must be even.
- If h^2 is even, so is *h*.
- Therefore *a* must be odd. (Since they cannot both be even.)



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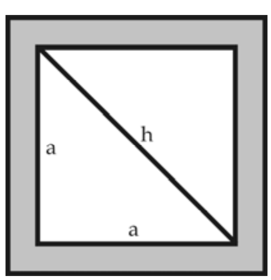
26

Proving Incommensurability, 4

- Since *h* is even, it is equal to 2 times some number, *j*. So $h = 2j$. Substitute $2j$ for *h* in the formula given by the Pythagorean theorem:
- $2a^2 = h^2 = (2j)^2 = 4j^2$.
- If $2a^2 = 4j^2$., then $a^2 = 2j^2$
- Therefore a^2 is even, and so is *a*.
- But we have already shown that *a* is odd.



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27

Proof by Contradiction

- This proof is typical of the use of logic, as championed by Parmenides, to sort what is true and what is false into separate categories.
- It is the cornerstone of Greek mathematical reasoning, and also is used throughout ancient reasoning about nature.

The Method of Proof by Contradiction

- 1. Assume the opposite of what you wish to prove:
  - Assume that the diagonal and the side are commensurable, meaning that at least one unit of length exists that exactly measures each.

The Method of Proof by Contradiction

- 2. Show that valid reasoning from that premise leads to a logical contradiction.
  - That the length of the side of the square must be both an odd number of units and an even number of units.
  - Since a number cannot be both odd and even, something must be wrong in the argument.
  - The only thing that could be wrong is the assumption that the lengths are commensurable.

The Method of Proof by Contradiction

- 3. Therefore the opposite of the assumption must be true.
  - If the only assumption was that the two lengths are commensurable and that is false, then it must be the case that the lengths are incommensurable.
  - Note that the conclusion logically follows even though at no point were any of the possible units of measure specified.

The Flaw of Pythagoreanism

- The Pythagorean world view – that everything that exists is ultimately a numerical structure (and that numbers mean just counting numbers—integers).
- In their greatest triumph, the magical Pythagorean theorem, lay a case that cannot fit this world view.

The Decline of the Pythagoreans

- The incommensurability of the diagonal and side of a square sowed a seed of doubt in the minds of Pythagoreans.
- They became more defensive, more secretive, and less influential.
- But they never quite died out.