Euclid

Axioms and Proofs

Logic at its Best

- Where Plato and Aristotle agreed was over the role of reason and precise logical thinking.
- Plato: From abstraction to new abstraction.
- Aristotle: From empirical generalizations to unknown truths.

Mathematical Reasoning

- Plato's Academy excelled in training mathematicians.
- Aristotle's Lyceum excelled in working out logical systems.
- They came together in a great mathematical system.

The Structure of Ancient Greek Civilization

- Ancient Greek civilization is divided into two major periods, marked by the death of Alexander the Great.

Hellenic Period

- From the time of Homer to the death of Alexander is the Hellenic Period, 800-323 BCE.
  - When the written Greek language evolved.
  - When the major literary and philosophical works were written.
  - When the Greek colonies grew strong and were eventually pulled together into an empire by Alexander the Great.

Hellenistic Period

- From the death of Alexander to the annexation of the Greek peninsula into the Roman Empire, and then on with diminishing influence until the fall of Rome.
- 323 BCE to 27 BCE, but really continuing its influence until the 5th century CE.

Science in the Hellenistic Age

- The great philosophical works were written in the Hellenic Age.
- The most important scientific works from Ancient Greece came from the Hellenistic Age.

Alexandria, Egypt

- Alexander the Great conquered Egypt, where a city near the mouth of the Nile was founded in his honour.
- Ptolemy Soter, Alexander's general in Egypt, established a great center of learning and research in Alexandria: *The Museum*.

The Museum

- The Museum – temple to the Muses – became the greatest research centre of ancient times, attracting scholars from all over the ancient world.
- Its centerpiece was the Library, the greatest collection of written works in antiquity, about 600,000 papyrus rolls.

Euclid

- Euclid headed up mathematical studies at the Museum.
- Little else is known about his life. He may have studied at Plato’s Academy.

Euclid’s *Elements*

- Euclid is now remembered for only one work, called *The Elements.*
- 13 “books” or volumes.
- Contains almost every known mathematical theorem, with logical proofs.

300 BCE – A Date to Remember

- You will have eight and only eight dates to remember in this course (although knowing more is helpful).
  - Each date is a marker of an important turning point in the development of science, for various reasons.
- This is the first one. It is the approximate **date of the publication of Euclid’s *Elements.***

The Influence of the *Elements*

- Euclid's *Elements* is the second most widely published book in the world, after the Bible.
- It was the definitive and basic textbook of mathematics used in schools up to the early 20th century.

Axioms

- What makes Euclid's *Elements* distinctive is that it starts with stated assumptions and derives all results from them, systematically.
- The style of argument is Aristotelian logic.
- The subject matter is Platonic forms.

Axioms, 2

- The axioms, or assumptions, are divided into three types:
  - Definitions
  - Postulates
  - Common notions
- All are assumed true.

Definitions

- The definitions simply clarify what is meant by technical terms. E.g.,
  - 1. A *point* is that which has no part.
  - 2. A *line* is breadthless length.
  - 10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands. …
  - 15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

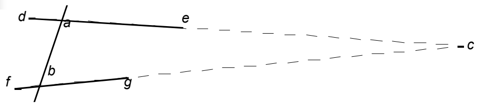
Postulates

- There are 5 postulates.
- The first 3 are "construction" postulates, saying that he will assume that he can produce (Platonic) figures that meet his ideal definitions:
  - 1. To draw a straight line from any point to any point.
  - 2. To produce a finite straight line continuously in a straight line.
  - 3. To describe a circle with any centre and distance.

Postulate 4

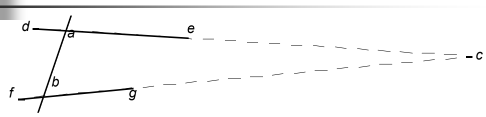
- 4. That all right angles are equal to one another.
- Note that the equality of right angles was not rigorously implied by the definition.
  - 10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*….
  - There could be other right angles not equal to these. The postulate rules that out.

The Controversial Postulate 5



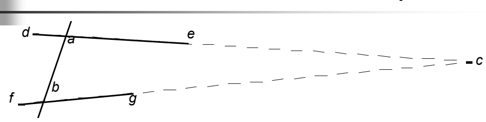
- 5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The "Parallel" Postulate



- One of Euclid's definitions was that lines are parallel if they never meet.
- Postulate 5, usually called the parallel postulate, gives a criterion for lines *not* being parallel.

The "Parallel" Postulate, 2



- This postulate is more like a mathematical theorem than an axiom, yet Euclid made it an assumption.
- For centuries, later mathematicians tried to prove the theorem from Euclid's other assumptions.

The Common Notions

- Finally, Euclid adds 5 "common notions" for completeness. These are really essentially logical principles rather than specifically mathematical ideas:
  - 1. Things which are equal to the same thing are also equal to one another.
  - 2. If equals be added to equals, the wholes are equal.
  - 3. If equals be subtracted from equals, the remainders are equal.
  - 4. Things which coincide with one another are equal to one another.
  - 5. The whole is greater than the part.

An Axiomatic System

- After all this preamble, Euclid is finally ready to prove some mathematical propositions.
- The virtue of this approach is that the assumptions are all laid out ahead. Nothing that follows makes further assumptions.

Axiomatic Systems

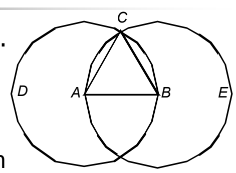
- The assumptions are clear and can be referred to.
- The deductive arguments are also clear and can be examined for logical flaws.
- The truth of any proposition then depends entirely on the assumptions and on the logical steps.
- And, the system builds. Once some propositions are established, they can be used to establish others.
  - Aristotle's methodology applied to mathematics.

The Propositions in the *Elements*

- For illustration, we will follow the sequence of steps from the first proposition of book I that lead to the 47th proposition of book I.
- This is more familiarly known as the Pythagorean Theorem.

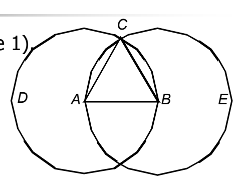
Proposition I.1 On a given finite straight line to construct an equilateral triangle.

- Let AB be the given line.
- Draw a circle with centre A having radius AB. (Postulate 3)
- Draw another circle with centre B having radius AB.
- Call the point of intersection of the two circles C.



Proposition I.1, continued

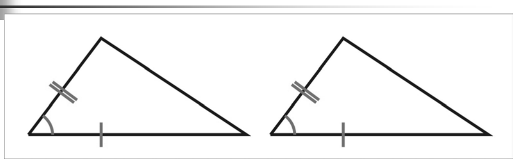
- Connect AC and BC (Postulate 1).
- AB and AC are radii of the same circle and therefore equal to each other (Definition 15, of a circle).
- Likewise AB=BC.
- Since AB=AC and AB=BC, AC=BC (Common Notion 1).
- Therefore triangle ABC is equilateral (Definition 20, of an equilateral triangle). Q.E.D.



What Proposition I.1 Accomplished

- Proposition I.1 showed that given only the assumptions that Euclid already made, he is able to show that he can construct an equilateral triangle on any given line. He can therefore use constructed equilateral triangles in other proofs without having to justify that they can be drawn all over again.
- Stories about Euclid:
  - No royal road.
  - Payment for learning.

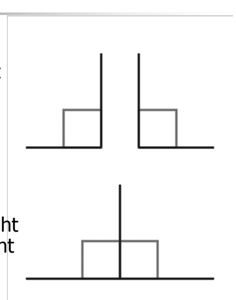
Other propositions that are needed to prove I.47



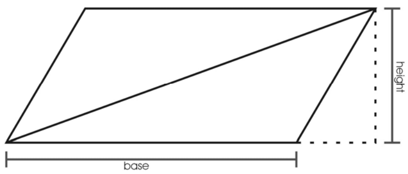
- Prop. I.4
  - If two triangles have two sides of one triangle equal to two sides of the other triangle plus the angle between the sides that are equal in each triangle is the same, then the two triangles are congruent

Other propositions that are needed to prove I.47

- Prop. I.14
  - Two adjacent right angles make a straight line.
  - Definition 10 asserted the converse, that a perpendicular erected on a straight line makes two right angles.



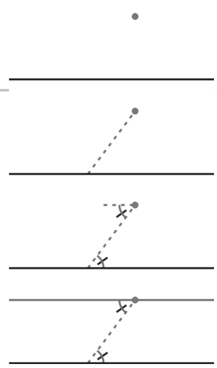
Other propositions that are needed to prove I.47



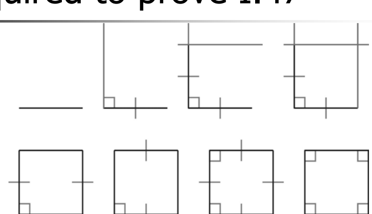
- Prop. I.41
  - The area of a triangle is one half the area of a parallelogram with the same base and height.

Constructions that are required to prove I.47

- Prop. I.31
  - Given a line and a point not on the line, a line through the point can be constructed parallel to the first line.



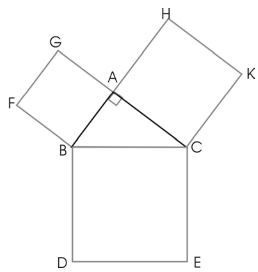
Constructions that are required to prove I.47



- Prop. I.46
  - Given a straight line, a square can be constructed with the line as one side.

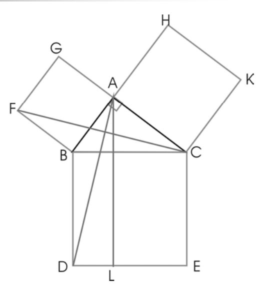
Proposition I.47

- In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

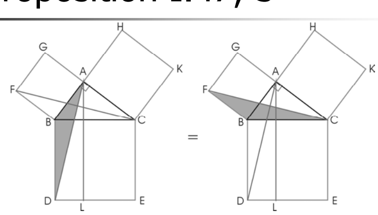


Proposition I.47, 2

- Draw a line parallel to the sides of the largest square, from the right angle vertex, A, to the far side of the triangle subtending it, L.
- Connect the points FC and AD, making ΔFBC and ΔABD.

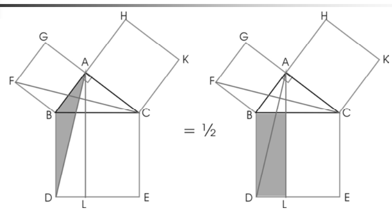


Proposition I.47, 3



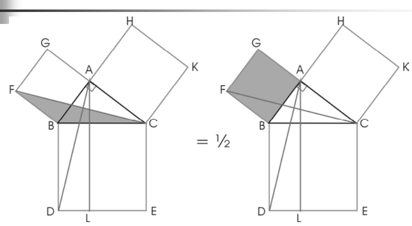
- The two shaded triangles are congruent (by Prop. I.4) because the shorter sides are respectively sides of the constructed squares and the angle between them is an angle of the original right triangle, plus a right angle from a square.

Proposition I.47, 4



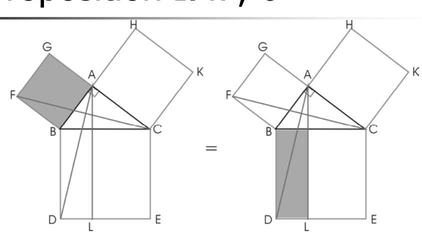
- The shaded triangle has the same base (BD) as the shaded rectangle, and the same height (DL), so it has exactly half the area of the rectangle, by Proposition I.41.

Proposition I.47, 5



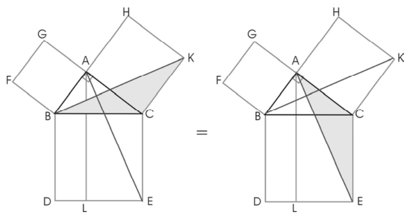
- Similarly, the other shaded triangle has half the area of the small square since it has the same base (FB) and height (GF).

Proposition I.47, 6



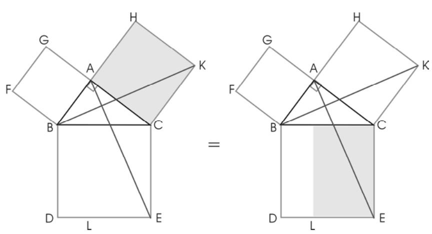
- Since the triangles had equal areas, twice their areas must also be equal to each other (Common notion 2), hence the shaded square and rectangle must also be equal to each other.

Proposition I.47, 7



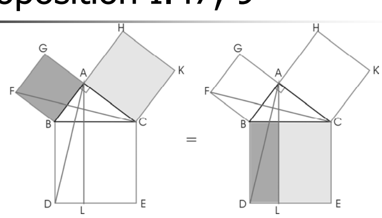
- By the same reasoning, triangles constructed around the other non-right vertex of the original triangle can also be shown to be congruent.

Proposition I.47, 8



- And similarly, the other square and rectangle are also equal in area.

Proposition I.47, 9



- And finally, since the square across from the right angle consists of the two rectangles which have been shown equal to the squares on the sides of the right triangle, those squares together are equal in area to the square across from the right angle.

Building Knowledge with an Axiomatic System

- Generally agreed upon premises ("obviously" true)
- Tight logical implication
- Proofs by:
  - 1. Construction
  - 2. Exhaustion
  - 3. *Reductio ad absurdum* (reduction to absurdity)
    - -- assume a premise to be true
    - -- deduce an absurd result

Example: Proposition IX.20

- There is no limit to the number of prime numbers
- Proved by
  - 1. Constructing a new number.
  - 2. Considering the consequences whether it is prime or not (method of exhaustion).
  - 3. Showing that there is a contraction if there is not another prime number. (*reduction ad absurdum).*

Proof of Proposition IX.20

- Given a set of prime numbers, $\{P_1,P_2,P_3,\ldots P_k\}$
- 1. Let $Q = P_1P_2P_3\ldots P_k + 1$ (Multiply them all together and add 1)
- 2. Q is either a new prime or a composite
- 3. If a new prime, the given set of primes is not complete.

- Example 1: {2,3,5}
- Q=2x3x5+1 =31
- Q is prime, so the original set was not complete. 31 is not 2, 3, or 5

- Example 2: {3,5,7}
- Q=3x5x7+1 =106
- Q is composite.

Proof of Proposition IX.20

- 4. If a composite, Q must be divisible by a prime number.
- -- Due to Proposition VII.31, previously proven.
- -- Let that prime number be G.
- 5. G is either a new prime or one of the original set, $\{P_1, P_2, P_3, \ldots P_k\}$
- 6. If G is one of the original set, it is divisible into $P_1P_2P_3 \ldots P_k$ If so, G is also divisible into 1, (since G is divisible into Q)
- 7. This is an absurdity.

- Q=106=2x53.
- Let G=2.
- G is a new prime (not 3, 5, or 7).
- If G was one of 3, 5, or 7, then it would be divisible into 3x5x7=105.
- But it is divisible into 106.
- Therefore it would be divisible into 1.
- This is absurd.

Proof of Proposition IX.20

- Follow the absurdity backwards.
- Trace back to assumption (line 6), that G was one of the original set. That must be false.
- The only remaining possibilities are that Q is a new prime, or G is a new prime.
- In any case, there is a prime other than the original set.
  - Since the original set was of arbitrary size, there is always another prime, no matter how many are already accounted for.