

Pythagoras

Numbers as the ultimate reality

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Pythagoras of Samos

- Born between 580 and 569. Died about 500 BCE.
- Lived in Samos, an island off the coast of Ionia.



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Pythagoras and the Pythagoreans

- Pythagoras himself lived earlier than many of the other Pre-Socratics and had some influence on them:
 - E.g., Heraclitos, Parmenides, and Zeno
- Very little is known about what Pythagoras himself taught, but he founded a cult that promoted and extended his views. Most of what we know is from his followers.

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The Pythagorean Cult

- The followers of Pythagoras were a close-knit group like a religious cult.
- Vows of poverty.
- Secrecy.
- Special dress, went barefoot.
- Strict diet:
 - Vegetarian
 - Ate no beans.

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Everything is Number

- The Pythagoreans viewed number as the underlying structure of everything in the universe.
 - Compare to Thales' view of water, Anaximander's *apeiron*, Anaximenes' air, Heraclitos, change.
- Pythagorean numbers take up space.
 - Like little hard spheres.

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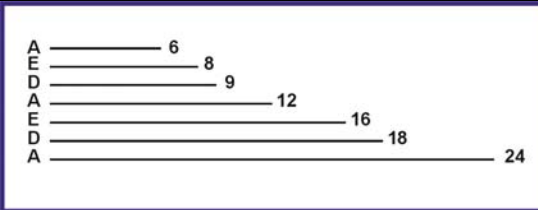
Numbers and Music

- One of the discoveries attributed to Pythagoras himself.
- Musical scale:
 - 1:2 = octave
 - 2:3 = perfect fifth
 - 3:4 = perfect fourth

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Numbers and Music, contd.



- Relative string lengths for notes of the scale from lowest note (bottom) to highest.
- The octave higher is half the length of the former. The fourth is $\frac{3}{4}$, the fifth is $\frac{2}{3}$.

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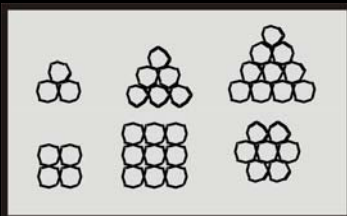
Geometric Harmony

- The numbers 12, 8, 6 represent the lengths of a ground note, the fifth above, and the octave above the ground note.
 - Hence these numbers form a "harmonic progression."
- A cube has 12 edges, 8 corners, and 6 faces.
 - Fantastic! A cube is in "geometric harmony."

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Figurate Numbers



- Numbers that can be arranged to form a regular figure (triangle, square, hexagon, etc.) are called figurate numbers.

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The Tetractys

- Special significance was given to the number 10, which can be arranged as a triangle with 4 on each side.
- Called the tetrad or *tetractys*.



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The significance of the Tetractys

- The number 10, the *tetractys*, was considered sacred.
- It was more than just the base of the number system and the number of fingers.
- The Pythagorean oath:
 - "By him that gave to our generation the Tetractys, which contains the fount and root of eternal nature."

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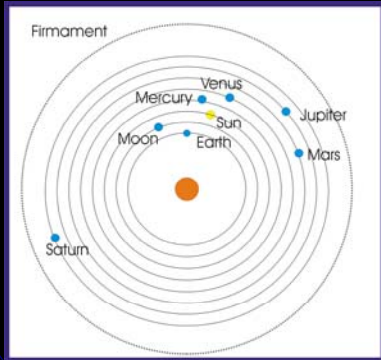
Pythagorean Cosmology

- Unlike almost every other ancient thinker, the Pythagoreans did not place the Earth at the centre of the universe.
- The Earth was too imperfect for such a noble position.
- Instead the centre was the "Central Fire" or, the watchtower of Zeus.

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The Pythagorean cosmos -- with 9 heavenly bodies

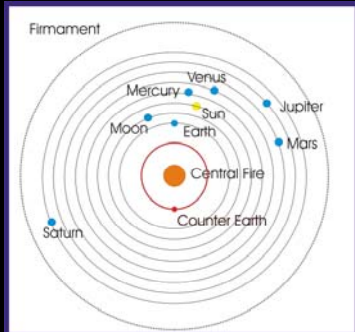


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The Pythagorean Cosmos and the Tetractys

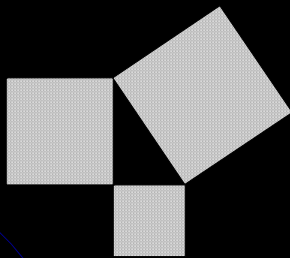
- To match the tetractys, another heavenly body was needed.
- Hence, the counter earth, or *antichthon*, always on the other side of the central fire, and invisible to human eyes.



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The Pythagorean Theorem

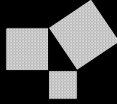


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The Pythagorean Theorem, contd.

- Legend has it that Pythagoras himself discovered the truth of the theorem that bears his name:



- That if squares are built upon the sides of *any* right triangle, the sum of the areas of the two smaller squares is equal to the area of the largest square.

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Well-known Special Cases

- Records from both Egypt and Babylonia as well as oriental civilizations show that special cases of the theorem were well known and used in surveying and building.
- The best known special cases are
 - The 3-4-5 triangle: $3^2+4^2=5^2$ or $9+16=25$
 - The 5-12-13 triangle: $5^2+12^2=13^2$ or $25+144=169$

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Commensurability

- Essential to the Pythagorean view that everything is ultimately number is the notion that the same scale of measurement can be used for everything.
- E.g., for length, the same ruler, perhaps divided into smaller and smaller units, will ultimately measure every possible length exactly.
- This is called *commensurability*.

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Commensurable Numbers

- Numbers, for the Pythagoreans, mean the natural, counting numbers.
- All natural numbers are commensurable because they can all be “measured” by the same unit, namely 1.
 - The number 25 is measured by 1 laid off 25 times.
 - The number 36 is measured by 1 laid off 36 times.

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Commensurable Magnitudes

- A magnitude is a measurable quantity, for example, length.
- Two magnitudes are commensurable if a common unit can be laid off to measure each one exactly.
 - E.g., two lengths of 36.2 cm and 171.3 cm are commensurable because each is an exact multiple of the unit of measure 0.1 cm.
 - 36.2 cm is exactly 362 units and 171.3 cm is exactly 1713 units.

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Commensurability is essential for the Pythagorean view.

- If everything that exists in the world ultimately has a numerical structure, and numbers mean some tiny spherical balls that occupy space, then everything in the world is ultimately commensurable with everything else.
- It may be difficult to find the common measure, but it just *must* exist.

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Incommensurability

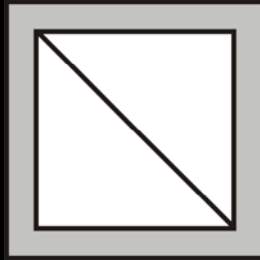
- The (inconceivable) opposite to commensurability is *incommensurability*, the situation where no common measure between two quantities exists.
- To prove that two quantities are commensurable, one need only find a single common measure.
- To prove that quantities are *incommensurable*, it would be necessary to prove that no common measures could possibly exist.

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The Diagonal of the Square

- The downfall of the Pythagorean world view came out of their greatest triumph the Pythagorean theorem.
- Consider the simplest case, the right triangles formed by the diagonal of a square.

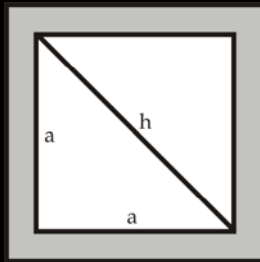


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Proving Incommensurability

- If the diagonal and the side of the square are commensurable, then they can each be measured by some common unit.
- Suppose we choose the largest common unit of length that goes exactly into both.

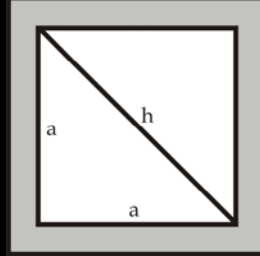


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Proving Incommensurability, 2

- Call the number of times the measuring unit fits on the diagonal h and the number of times it fits on the side of the square a .
- It cannot be that a and h are both even numbers, because if they were, a larger unit (twice the size) would have fit exactly into both the diagonal and the side.

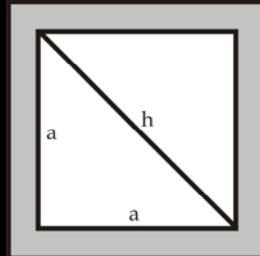


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Proving Incommensurability, 3

- By the Pythagorean theorem, $a^2 + a^2 = h^2$
- If $2a^2 = h^2$ then h^2 must be even.
- If h^2 is even, so is h .
- Therefore a must be odd. (Since they cannot both be even.)

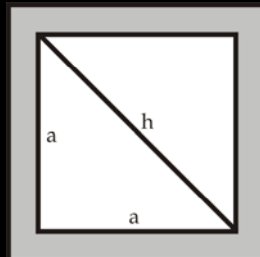


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Proving Incommensurability, 4

- Since h is even, it is equal to 2 times some number, j . So $h = 2j$. Substitute $2j$ for h in the formula given by the Pythagorean theorem:
- $2a^2 = h^2 = (2j)^2 = 4j^2$.
- If $2a^2 = 4j^2$, then $a^2 = 2j^2$
- Therefore a^2 is even, and so is a .
- But we have already shown that a is odd.



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Proof by Contradiction

- This proof is typical of the use of logic, as championed by Parmenides, to sort what is true and what is false into separate categories.
- It is the cornerstone of Greek mathematical reasoning, and also is used throughout ancient reasoning about nature.

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The Method of Proof by Contradiction

1. Assume the opposite of what you wish to prove:
 - Assume that the diagonal and the side are commensurable, meaning that at least one unit of length exists that exactly measures each.

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The Method of Proof by Contradiction

2. Show that valid reasoning from that premise leads to a logical contradiction.
 - That the length of the side of the square must be both an odd number of units and an even number of units.
 - Since a number cannot be both odd and even, something must be wrong in the argument.
 - The only thing that could be wrong is the assumption that the lengths are commensurable.

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The Method of Proof by Contradiction

- 3. Therefore the opposite of the assumption must be true.
 - If the only assumption was that the two lengths are commensurable and that is false, then it must be the case that the lengths are incommensurable.
 - Note that the conclusion logically follows even though at no point were any of the possible units of measure specified.

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The Flaw of Pythagoreanism

- The Pythagorean world view – that everything that exists is ultimately a numerical structure (and that numbers mean just counting numbers—integers).
- In their greatest triumph, the magical Pythagorean theorem, lay a case that cannot fit this world view.

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The Decline of the Pythagoreans

- The incommensurability of the diagonal and side of a square sowed a seed of doubt in the minds of Pythagoreans.
- They became more defensive, more secretive, and less influential.
- But they never quite died out.

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