2/13/13 ← NOTE: date shift due to snow cancellation on 2/8

Ampère's Law (cont)

- Similar in spirit to Gauss' Law to determine $\bar{E}$, this rule allows us to exploit symmetries to find $\bar{B}$

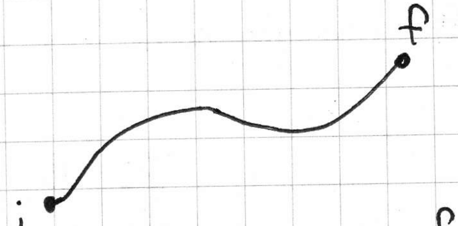
$$\sum_{\substack{\text{closed}\\\text{path}}} B_{\parallel}\,\Delta L = \mu_0 I_{enclosed}$$ (Giordano version)

we can alternatively write this as:

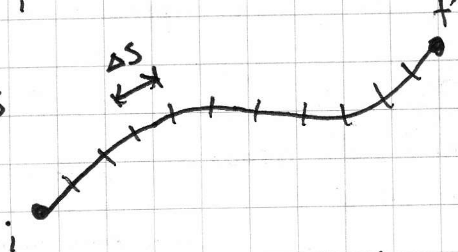
$$\oint_C \bar{B}\cdot d\bar{s} = \mu_0 I_{enclosed}$$ (NOTE $I_{enclosed} = I_{through}$)

here $\oint_C$ indicates a 'line integral' (i.e. integration is performed around a **closed curve**). The integrand dot product is a scalar that represents a specific relationship between $\bar{B}$ and the curve of interest.

Math Aside (re line integrals)



- Consider an arbitrary line (that doesn't cross itself) stretched between i and f (two points)



- chop that line up into K little segments of length ΔS
- the length of the line is then

$$l = \sum_K \Delta s_K \longrightarrow \int_i^f ds$$

→ curve has total length l

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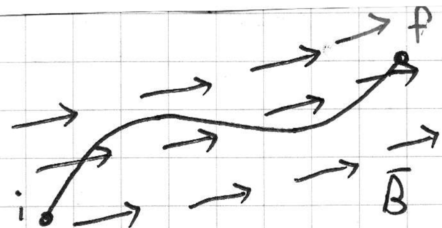
where $ds = \lim_{\Delta s \to 0} \Delta s$

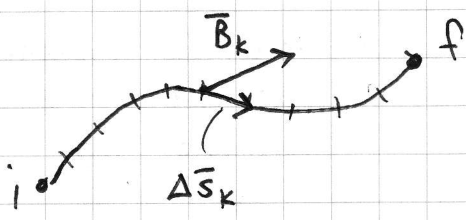
- this last piece is what is called a line integral, i.e. we chopped the line up into infinitely many infinitesimal pieces, then added them up

  ex $\int_a^b dx = x\big|_a^b = b-a$ is just a line integral along the x-axis from b to a!

- Now what happens if we make things a bit more interesting and consider our line relative to a magnetic field $\bar{B}$?



  - the k'th element along the line has length Δs, but also has a direction associated with it (tangent to the line), the $\Delta \bar{s}_k$ is a vector



- we can also consider the magnetic field at that point ($\bar{B}_k$) which is also a vector

- Suppose we want to compute the dot product $\bar{B}_k \cdot \Delta \bar{s}_k$ at each segment and then add them up along the line (this is what we need to do for Ampère's Law!), then we have:

$$\sum_k \bar{B}_k \cdot \Delta \bar{s}_k \longrightarrow \int_i^f \bar{B} \cdot d\bar{s} = \text{line integral of } \bar{B} \text{ from } i \text{ to } f$$

- If we make use of symmetry, this calculation is easy:
  - if $\bar{B}$ is perpendicular everywhere to the line, $\bar{B} \cdot d\bar{s} = 0$ and integral is zero
  - if $\bar{B}$ is parallel (i.e. tangent) to the curve everywhere and B is const. along the line, then (cont ↓)

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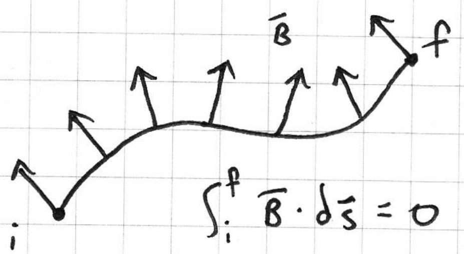
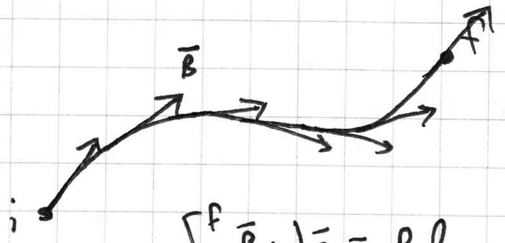
(cont.) $\bar{B}\cdot d\bar{s} = B\,ds$ and $\int_i^f \bar{B}\cdot d\bar{s} = B\int_i^f ds = Bl$

→ we can also show this visually



$\int_i^f \bar{B}\cdot d\bar{s} = 0$



$\int_i^f \bar{B}\cdot d\bar{s} = Bl$

- Now back to Ampère's Law: $\oint_c \bar{B}\cdot d\bar{s} = \mu_o I_{enclosed}$
  - $\oint \bar{B}\cdot d\bar{s}$ indicates our curve is a 'closed' one (i.e. the starting point and ending point are the same); put another way, we integrate around a loop
  - $I_{enclosed}$ is the total current passing through the surface that the closed curve enclosed
    (Aside: sometimes $I_{enclosed}$ is written as

$$I_{enclosed} = \iint_s \bar{J}\cdot d\bar{s}$$ where $\bar{J}$ is the 'current density' and $\iint_s$ indicates integrating over the surface S that is enclosed by C; Gauss' Law is written in a similar fashion in its general form

- Like Gauss' Law, $\oint_c \bar{B}\cdot d\bar{s} = \mu_o I$ has the following properties:
  - the integral is independent of the shape around the current
  - its independent of where the current passes through
  - depends only upon the total amount of current passing through the integration paths enclosing area

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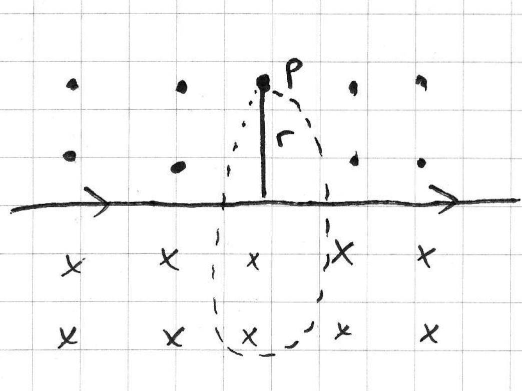
→ For HW10, for this to be generally useful, we need to apply it to case where we can find a path along which $\bar{B}$ is const. (these are few and far between!)

ex a straight wire w/ current I

- path is a circle of radius r around the wire ($\bar{B}$ is const. along this circle!)



$$\oint_c \bar{B}\cdot d\bar{s} = B\oint_c ds$$

$$= B(2\pi r) = \mu_0 I \longrightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

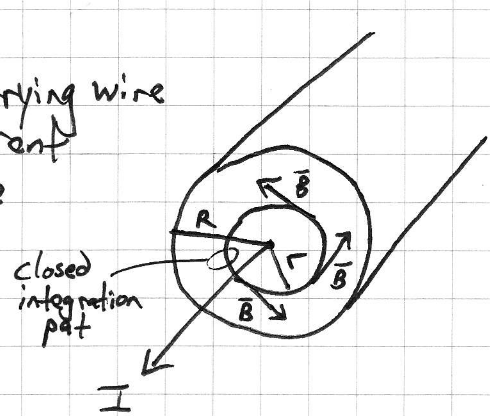
- as shown in ch. 20.7 by Giordano, for a 1 A current, 1 cm away from the wire yields:

$$B = \frac{(4\pi\times10^{-7}\ T\cdot m/A)(1\ A)}{2\pi\,(0.01\ m)} = 2\times10^{-5}\ T$$

→ 1 A is a large current and 0.01 m is pretty close to the wire, yet the field strength is quite small! (not even ½ that of earth)

ex Magnetic field inside a current-carrying wire of radius R. Assume the current density ($\bar{J}$) is uniform over the cross-section of the wire.

- make use of circular symmetry! (see right)



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ex. (cont.)

- Consider a circle of radius r ($r < R$). The current passing through is:

$$I_{through} = JA = J\pi r^2$$

- We assumed the current density is uniform, so we can determine J in terms of I and R:

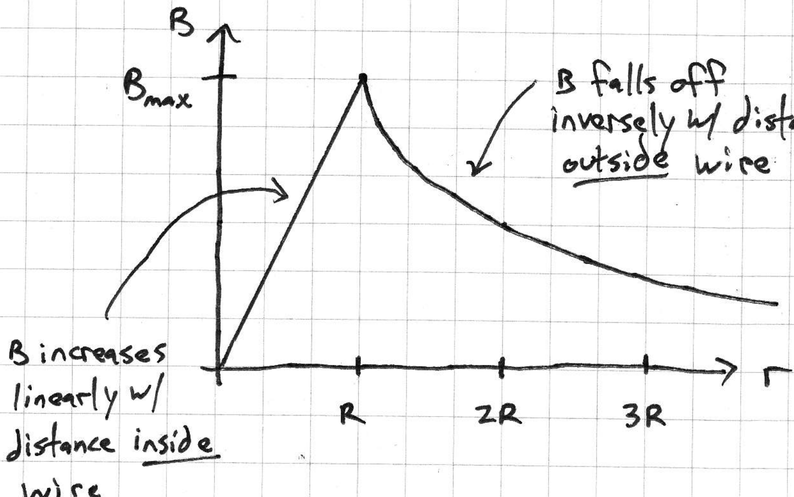
$$J = \frac{I}{A} = \frac{I}{\pi R^2} \longrightarrow I_{through} = \frac{r^2}{R^2} I$$

- Now $\bar{B}$ is uniform and tangent to our circle, so we have:

$$\oint \bar{B} \cdot d\bar{s} = B\,2\pi r = \mu_0 I_{through} = \mu_0 \left(\frac{r^2}{R^2}\right) I$$

- Now we can solve for B:

$$B = \frac{\mu_0 I}{2\pi R^2} r$$



$$B_{max} = \frac{\mu_0 I}{2\pi R}$$

(i.e. B is largest on the surface of the wire)

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- Finding B for a loop of coil is trickier and beyond the scope of 1410. We will simply state the conclusion (as well as that for a coil of N loops) and provide the derivation via the Biot-Savart law below for reference (you won't be responsible for this for an exam)
[you do need to know the two directly below though!]

- Single ~~coil~~ loop of wire w/ current I:
(radius R, at center of loop)

$$B_{loop} = \frac{\mu_0 I}{2R}$$

- coil of N loops of radius R w/ current I (at center of coil):

$$B_{coil} = N\frac{\mu_0 I}{2R}$$

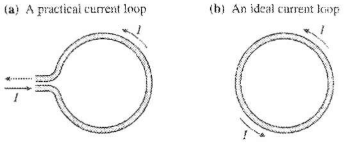
(they simply sum up together!)

The magnetic field of a current loop

FIGURE 32.16a shows a current loop, a circular loop of wire with radius R that carries current I. Find the magnetic field of the current loop at distance z on the axis of the loop.

FIGURE 32.16 A current loop.

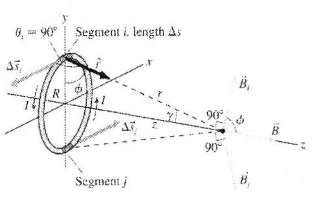
(a) A practical current loop (b) An ideal current loop



Real coils need wires to bring the current in and out, but we'll model the coil as a current moving around the full circle shown in FIGURE 32.16b.

FIGURE 32.17 shows a loop for which we've assumed that the current is circulating ccw. We've chosen a coordinate system in which the loop lies at $z = 0$ in the xy-plane. Let segment i be the segment at the top of the loop. Vector $\Delta\vec{s}_i$ is parallel to the x-axis and unit vector $\hat{r}$ is in the yz-plane, thus angle θ_i, the angle between $\Delta\vec{s}_i$ and $\hat{r}$, is 90°.

FIGURE 32.17 Calculating the magnetic field of a current loop.



The direction of $\vec{B}_i$, the magnetic field due to the current in segment i, is given by the cross product $\Delta\vec{s}_i \times \hat{r}$. $\vec{B}_i$ must be perpendicular to $\Delta\vec{s}_i$ *and* perpendicular to $\hat{r}$. You should convince yourself that $\vec{B}_i$ in Figure 32.17 points in the correct direction. Notice that the y-component of $\vec{B}_i$ is canceled by the y-component of magnetic field $\vec{B}_j$ due to the current segment at the bottom of the loop, 180° away. In fact, *every* current segment on the loop can be paired with a segment 180° away, on the opposite side of the loop, such that the x- and y-components of $\vec{B}$ cancel and the components of $\vec{B}$ parallel to the z-axis add. In other words, the symmetry of the loop requires the on-axis magnetic field to point along the z-axis. Knowing that we need to sum only the z-components will simplify our calculation.

We can use the Biot-Savart law to find the z-component $(B_i)_z = B_i\cos\phi$ of the magnetic field of segment i. The cross product $\Delta\vec{s}_i \times \hat{r}$ has magnitude $(\Delta s)(1)\sin 90° = \Delta s$, thus

$$(B_i)_z = \frac{\mu_0}{4\pi}\frac{I\,\Delta s}{r^2}\cos\phi = \frac{\mu_0 I\cos\phi}{4\pi(z^2+R^2)}\Delta s$$

where we used $r = (z^2+R^2)^{1/2}$. You can see, because $\phi + \gamma = 90°$, that angle ϕ is also the angle between $\hat{r}$ and the radius of the loop. Hence $\cos\phi = R/r$, and

$$(B_i)_z = \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}\Delta s$$

The final step is to sum the magnetic fields due to all the segments:

$$B_{loop} = \sum_i (B_i)_z = \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}\sum_i \Delta s$$

In this case, unlike the straight wire, none of the terms multiplying Δs depends on the position of segment i, so all these terms can be factored out of the summation. We're left with a summation that adds up the lengths of all the small segments. But this is just the total length of the wire, which is the circumference $2\pi R$. Thus the on-axis magnetic field of a current loop is

$$B_{loop} = \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}2\pi R = \frac{\mu_0}{2}\frac{IR^2}{(z^2+R^2)^{3/2}}$$

Note that at the center of the coil, $z = 0$ and the general expression reduces to B_{loop}

from Knight