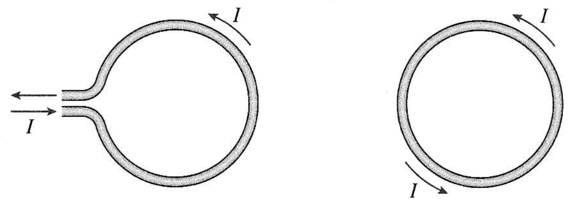
EXAMPLE 32.5 The magnetic field of a current loop

FIGURE 32.16a shows a current loop, a circular loop of wire with radius R that carries current I. Find the magnetic field of the current loop at distance z on the axis of the loop.

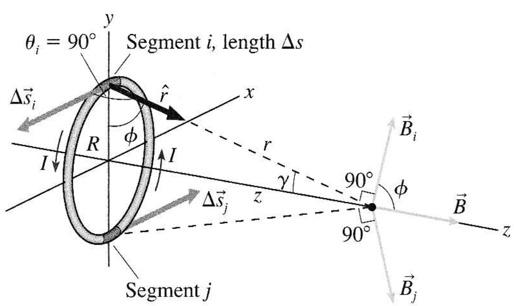
FIGURE 32.16 A current loop.



MODEL Real coils need wires to bring the current in and out, but we'll model the coil as a current moving around the full circle shown in FIGURE 32.16b.

VISUALIZE FIGURE 32.17 shows a loop for which we've assumed that the current is circulating ccw. We've chosen a coordinate system in which the loop lies at $z = 0$ in the xy-plane. Let segment i be the segment at the top of the loop. Vector $\Delta\vec{s}_i$ is parallel to the x-axis and unit vector $\hat{r}$ is in the yz-plane, thus angle θ_i, the angle between $\Delta\vec{s}_i$ and $\hat{r}$, is 90°.

FIGURE 32.17 Calculating the magnetic field of a current loop.



The direction of $\vec{B}_i$, the magnetic field due to the current in segment i, is given by the cross product $\Delta\vec{s}_i \times \hat{r}$. $\vec{B}_i$ must be perpendicular to $\Delta\vec{s}_i$ *and* perpendicular to $\hat{r}$. You should convince yourself that $\vec{B}_i$ in Figure 32.17 points in the correct direction. Notice that the y-component of $\vec{B}_i$ is canceled by the y-component of magnetic field $\vec{B}_j$ due to the current segment at the bottom of the loop, 180° away. In fact, *every* current segment on the loop can be paired with a segment 180° away, on the opposite side of the loop, such that the x- and y-components of $\vec{B}$ cancel and the components of $\vec{B}$ parallel to the z-axis add. In other words, the symmetry of the loop requires the on-axis magnetic field to point along the z-axis. Knowing that we need to sum only the z-components will simplify our calculation.

SOLVE We can use the Biot-Savart law to find the z-component $(B_i)_z = B_i \cos\phi$ of the magnetic field of segment i. The cross product $\Delta\vec{s}_i \times \hat{r}$ has magnitude $(\Delta s)(1)\sin 90° = \Delta s$, thus

$$(B_i)_z = \frac{\mu_0}{4\pi}\frac{I\,\Delta s}{r^2}\cos\phi = \frac{\mu_0 I\cos\phi}{4\pi(z^2+R^2)}\Delta s$$

where we used $r = (z^2 + R^2)^{1/2}$. You can see, because $\phi + \gamma = 90°$, that angle ϕ is also the angle between $\hat{r}$ and the radius of the loop. Hence $\cos\phi = R/r$, and

$$(B_i)_z = \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}\Delta s$$

The final step is to sum the magnetic fields due to all the segments:

$$B_{\text{loop}} = \sum_i (B_i)_z = \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}\sum_i \Delta s$$

In this case, unlike the straight wire, none of the terms multiplying Δs depends on the position of segment i, so all these terms can be factored out of the summation. We're left with a summation that adds up the lengths of all the small segments. But this is just the total length of the wire, which is the circumference $2\pi R$. Thus the on-axis magnetic field of a current loop is

$$B_{\text{loop}} = \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}2\pi R = \frac{\mu_0}{2}\frac{IR^2}{(z^2+R^2)^{3/2}}$$