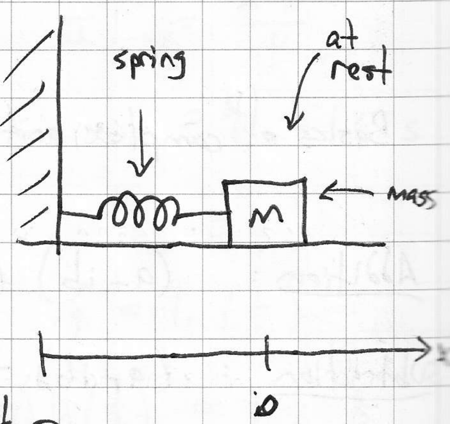
3/13/13 **Harmonic Oscillator** (see Giordano ch. 11.2 ff)

- Let's consider a basic (but very important!) system, a **mass-on-a-spring**

- Some intial thoughts before we dig in:



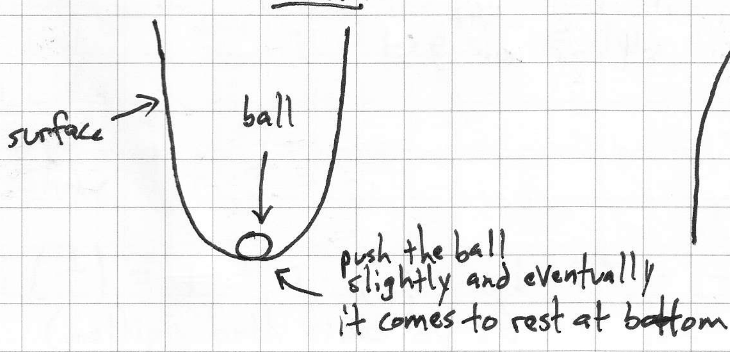
  - If we displace the mass, it will oscillate. Eventually the oscillations will die down and the system will come to rest

  → we call this a damped harmonic oscillator (When there is no damping, this is called **simple harmonic motion**)

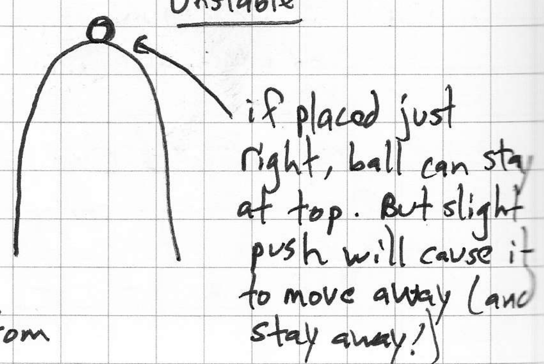
  - There is some **characteristic frequency** the system will oscillate at (we'll come back to this in a bit)

  - In terms of coming to rest, this motivates the notion of an **equilibrium point**. This is an important concept. Most generally, such a point can be stable or unstable

Stable



Unstable



3/13/13

→ An equilibrium point is such that the state of a system doesn't change when plased there w/ the right conditions. But perturb it a bit, and the stability of the point will dictate the subsequent behavior

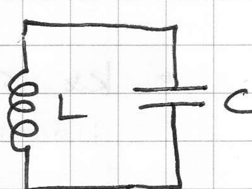
- Energy: Kinetic (mass moving) vs Potential (stretched/compressed spring)

  → energy is transferred back + forth!

- Very similar to other physical cases such as a (simple) pendulum and torsional balance

  → remarkably, a simple harmonic oscillator is directly analogous to an LC circuit

(we'll come back to this in ch. 22, but as a proview: L ⟷ mass, C ⟷ spring)



- In terms of energy (which will soon described in detail), the mass/spring when driven by a external force (e.g. a sinusoid) gives rise to the notion of **resonance** (this ties back to the characteristic frequency)

- As we'll see, it is very helpful to think about things via motion around a circle (see 3/11 notes) in addition to the eqns. of motion

3/13/13

- Our starting point is Hooke's Law and Newton's 2nd law:

$$F = m\ddot{x} = -kx$$

Note: $\ddot{x} \equiv \frac{d^2x}{dt^2} \equiv a$ ← acceleration

m - mass of 'mass' (e.g. block)
k - spring constant [N/m] → stiffer spring = larger k
x - position of mass
(let $x=0$ be the equilibrium, $x<0$ means the spring is compressed, $x>0$ spring is stretched)

- $m\ddot{x}$ is the force ~~due~~ associated w/ the objects momentum
- kx is the force associated w/ the stretched/compressed spring

NOTE $F = -kx$ (not $+kx$) [Hooke's Law]

→ Just like Lenz's law for induction, the sign matters! The force is restorative

Note: this is a linear 2nd order ODE!

- Let's rewrite things a bit: $\frac{d^2x}{dt^2} = -\frac{k}{m}x \equiv -\omega_0^2 x$

where $\omega_0 \equiv \sqrt{k/m}$

→ Now, what sort of function, when differentiated twice, returns to its same form but negative?

⟹ a sinusoid!!

∘ let $x(t) = A\cos(\omega_0 t + \phi)$

then

NOTE
A – amplitude
$\theta \equiv \omega_0 t + \phi$ – phase

$v = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$

velocity

$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = -A\omega_0^2 \cos(\omega_0 t + \phi) = -\omega_0^2 x$

⇒ so $x(t) = A\cos(\omega_0 t + \phi)$ satisfies our eqn. of motion where A and ϕ will be determined the initial conditions

[NOTE: We could have chosen $x(t) = A\sin(\omega_0 t + \phi)$ as well!]

□ So ω_0, ~~the~~ deserves special mention:

- this (ω_0) is the characteristic frequency and is determined by k and m (small/stiff things tend to oscillate rapidly while big/pliant things oscillate more slowly)
- in terms of frequency: $f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{k/m}$ [Hz]

□ Let's consider the energy of the system:

potential energy stored in spring is $U = \frac{1}{2}kx^2$ (see Giordano ch. 6.4)

$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

$= \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

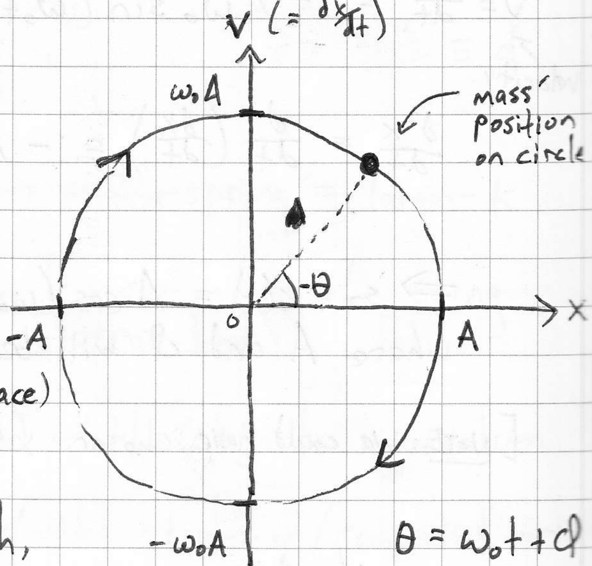
$= \frac{1}{2}kA^2 \sin^2(\omega_0 t + \phi) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$ [since $m\omega_0^2 = k$]

$= \frac{1}{2}kA^2$ [since $\cos^2\theta + \sin^2\theta = 1$]

3/13/13

→ so the total amount of energy is constant and depends only upon k (not m), which makes sense when you think about it

- Consider visualizing the motion of the mass as moving around an **ellipse** whose horizontal axis is the position of the mass and vertical axis is the velocity (dx/dt) (NOTE: this is called phase space)



$\theta = \omega_0 t + \phi$

- As mass oscillates back and forth, it traces out the ellipse is a clock-wise fashion (the direction matters!) for this choice of axes

- When $x = 0$, $U = 0$ and K is maximal. When $v = 0$, $K = 0$ and U is maximal

→ Energy is essentially just getting traded back and forth between K and U. Or put another way, energy is getting shifted between the mass (due to it's velocity) and spring (due to it compression). This transfer back and forth is the basis for the oscillation.

Looking Ahead:
1) What is there is some damping? → amplitude no longer const, but decays
2) Sinusoidal forcing? → system reaches a steady-state that depends upon freq. of forcing