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Damped / Forced Oscillations $\rightarrow$ Resonance

- What happens when there is damping (e.g. due to friction between the mass and the surface)?

drag force $\propto$ velocity (i.e. $\dot{x} = \frac{dx}{dt} = v$)

$$\rightarrow \quad m\ddot{x} = -kx - \underbrace{b\dot{x}}$$

- negative because it resists the direction of motion
- b - damping coefficient (eg. coefficient of friction)

Rewrite: $\quad \ddot{x} = -\omega_0^2 x - \gamma\dot{x} \qquad \left(\gamma \equiv \frac{b}{m}\right)$

- Clearly $x(t) = A\cos(\omega_0 t + \phi)$ no longer works (try plugging it in!). Intuitively, this makes sense since the amplitude must ~~no~~ longer remain const. but decrease w/ time since energy is being lost.

- Solving this eqn. is fairly straightforward, but beyond the scope of 1410. Nonetheless, it is worthwhile to highlight two methods and their results

1) Assume $x(t)$ has the form: $\quad x(t) = \underbrace{A_0}_{\text{starting amplitude}} \underbrace{e^{-t/\tau}}_{\text{assumes amplitude decrease exponentially w/ some rate const.}} \cos\underbrace{(\omega t}_{\text{Note: we don't assume the oscillation occurs at } \omega_0!} + \phi)$

$\rightsquigarrow$ we can plug this into our eqn. of motion (it gets a bit messy, but manageable)

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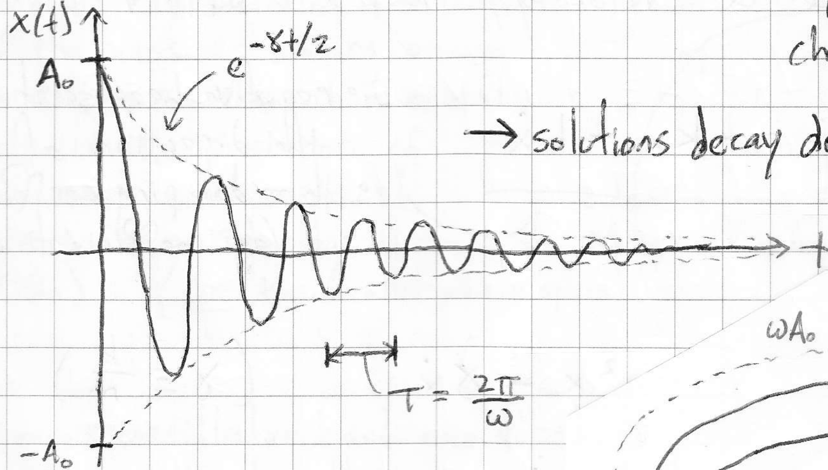
1) [cont.] When the smoke clears, we are left with

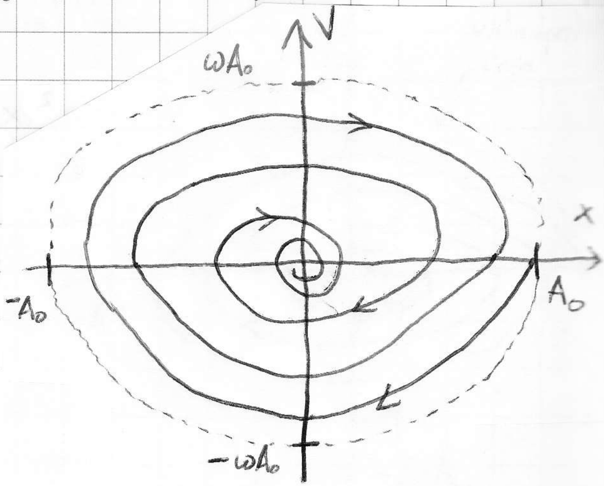
$$x(t) = A_0 e^{-\gamma t/2} \cos(\omega t + \phi)$$

where $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{2m}$ (mass oscillates at slightly lower freq. than the characteristic freq. ω_0)



→ solutions decay down to zero

→ in phase space, the 'trajectory' spirals down into the origin



- Note that depending upon the value of γ, the damping can be so fast that oscillations may not even occur (see Giordano Fig. 11.26)

⇒ this transition is called critical damping and occurs when $\gamma = 2\omega_0$

($\gamma < 2\omega_0 \rightarrow$ underdamped, $\gamma > 2\omega_0 \rightarrow$ overdamped)

2) This problem is very easy using complex numbers! (since differentiating an exponential is easy)

Assume $z(t) = A_o e^{i(pt+\phi)}$ where $p = n + is$

$= A_o e^{-st} e^{i(nt+\phi)} = x + iy$ [i.e. $x(t) = A_o e^{-st}\cos(nt+\phi)$ since $e^{i\theta} = \cos\theta + i\sin\theta$]

⇝ so $z(t)$ contains our assumed solution form for $x(t)$, just expresses it in a different way!!

- Not $\frac{dz}{dt} = ip\, A_o e^{i(pt+\phi)} = ipz$

$$\frac{d^2z}{dt^2} = -p^2 A_o e^{i(pt+\phi)} = -p^2 z \quad (\text{since } i \cdot i = -1)$$

So then $\frac{d^2z}{dt^2} + \gamma \frac{dz}{dt} + \omega_o^2 z = 0$ simply becomes

$$(-p^2 + ip\gamma + \omega_o^2)\underbrace{A_o e^{i(pt+\phi)}}_{\text{this is only zero for the 'trivial solution'}} = 0 \rightarrow -p^2 + ip\gamma + \omega_o^2 = 0$$

Aside: this is essentially an expression for the system's eigenvalues

Since $p = n + is$ and $p^2 = n^2 + 2ins - s^2$, we have

$$-n^2 - 2ins + s^2 + in\gamma - s\gamma + \omega_o^2 = 0 \longrightarrow \begin{array}{l} -n^2 + s^2 - s\gamma + \omega_o^2 = 0 \\ -2ns + n\gamma = 0 \end{array}$$

(equate real and imaginary parts)

So then $s = \frac{\gamma}{2}$ and plugging that in reveals $n^2 = \omega_o^2 - \frac{\gamma^2}{4}$

⇒ which is essentially what we already determined!!

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☐ Damped oscillator Summary: While a bit too sophisticated mathematically for 1410 (i.e. you will not be tested on the derivation shown on the last two pgs. on notes) it is important you understand the associated physical concepts:

- Damping causes loss of energy from the system

$$E = K + U = \tfrac{1}{2} m v^2(t) + \tfrac{1}{2} k x^2(t) = E(t)$$

→ $x(t)$ and $v(t)$ (and thus $E(t)$!) are decreasing functions due to loss of energy via damping

- The degree of damping affects whether oscillations even occur in the first place (i.e. underdamped vs. critically damped vs. overdamped)

→ presence of damping plays a crucial role in the behavior of the system!

☐ Let's consider another case: the **driven** (undamped) oscillator

⇝ basic idea is that there is an external sinusoidal driving force that can put energy into the system on a cycle-by-cycle basis

ex An empty soda bottle can actually be described by a harmonic oscillator (try looking up **Helmholtz resonator** on wikipedia). When you blow across the top of the bottle, you are the 'external' driving force

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- We modify our eqn. of motion to have to form:

$$m\ddot{x} = -kx + \underbrace{F_0 \cos \omega t}$$ (no damping)

we drive the system sinusoidally at frequency ω (i.e. we have control over ω)

- Assume the system has been driven for a long time such that it is at 'steady-state'. Then we assume a 'solution' of the form:

$$x(t) = A\cos(\omega t)$$

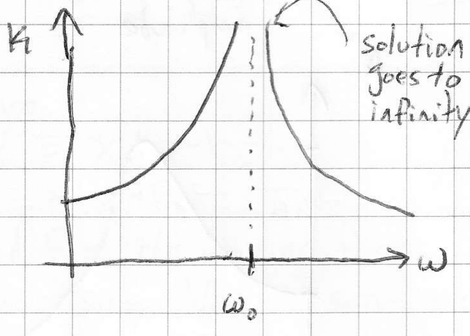
then $-m\omega^2 B\cos(\omega t) + kB\cos(\omega t) = F_0\cos(\omega t)$

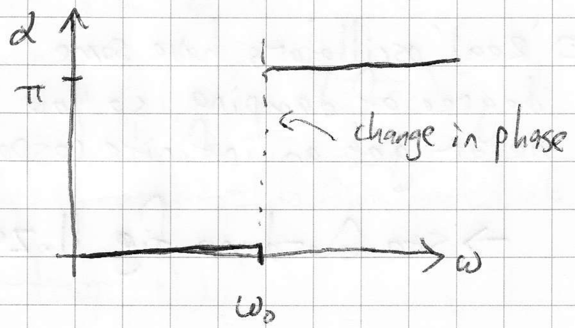
$$\rightarrow B = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

- So our solution has the form: $x(t) = \dfrac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t)$

or we rewrite as $x(t) = K_1\cos(\omega t + \alpha)$

where $K_1 = |B|$ and $\alpha = \begin{cases} 0 & \omega < \omega_0 \\ \pi & \omega > \omega_0 \end{cases}$





$\rightarrow$ this tells us a lot!! (next pg.)

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- when the system is driven close to ω_0, we get a big response. the farther away ω is from ω_0, the smaller the response

→ this is ~~the~~ basis #1 for **resonance**
(i.e. the system is 'tuned')

- if $\omega = \omega_0$, the 'steady-state' response is infinite. Physically this actually makes sense! The external force puts energy into the system on a cycle-by-cycle basis. When $\omega = \omega_0$, 100% of this energy is stored (via a perfect transfer of energy between the spring and mass). Given this perfectly efficient storage of energy, the system just keeps absorbing energy!

→ this is basis #2 for **resonance**
(i.e. the mass and spring effectively transfer energy back and forth)

- The oscillator experiences a phase change about $\omega = \omega_0$ (i.e. it goes from being in phase w/ the stimulus to being ½ a cycle out of phase)

- 'Real' oscillators have some degree of damping, so you never get an infinite response

→ see Giordano fig. 11.29