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SOLUTIONS

1 a) At the maxima, all energy is stored as potential energy in the spring as given by:

$$U = \tfrac{1}{2}kx^2$$

Initially, $x_0 \simeq 3$ (dimension-less). At the next full cycle (i.e. the successive max. displacement), $x_1 \simeq 1.5$.

So $$U_1/U_0 = \frac{\frac{1}{2}kx_1^2}{\frac{1}{2}kx_0^2} = \frac{(1.5)^2}{(3)^2} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$\Rightarrow$ the fraction of energy lost per cycle is about 75%

b) $x_2 \simeq 0.8$, so $U_2/U_1 \simeq 0.28$ (within our error of estimating x_2). Similarly for x_3 and so. Thus everytime the system oscillates, it will lose ~75% of the remaining energy.

Another way to determine this more precisely is as follows:

$$A(t) = A_0 e^{-t/\tau} \quad \left(\frac{1}{\tau} = \frac{\gamma}{2} \text{ where } \gamma = \frac{b}{m}\text{; see 3/15 notes}\right)$$

Consider two times t_1 and t_2 ($t_2 > t_1$). Then

$$\frac{A(t_2)}{A(t_1)} = \frac{e^{-t_2/\tau}}{e^{-t_1/\tau}} = e^{\frac{1}{\tau}(t_1 - t_2)} = e^{-\Delta t/\tau}$$

when $\Delta t = t_2 - t_1$. So for any fixed Δt, the ratio of the amplitudes will be the same, regardless of the actual values of t_1 and t_2 (only the difference matters!)

c) System is underdamped, since it is oscillating.

$\therefore$

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$\boxed{2}$ a) Conservation of energy of m_1: $K = U$ (all potential energy due to drop becomes kin-

$$\frac{1}{2} m_1 V_1^2 = m_1 g h \quad \rightarrow \quad V_1 = \sqrt{2gh} = 6.3 \text{ m/s}$$

Upon collision, momentum must be conserved:

$$m_1 V_1 = (m_1 + m_2) V_{12}$$ (where V_{12} is the initial velocity of the combined mass system)

$$\rightarrow V_{12} = \frac{m_1 V_1}{m_1 + m_2} = \frac{(3.4)(6.3)}{(3.4 + 1.1)} = 4.76 \text{ m/s}$$

b) $$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m_{total}}} = \frac{1}{2\pi}\sqrt{\frac{K}{m_1 + m_2}} = 5.3 \text{ Hz}$$

c) We can take the displacement of the combined system to be described by:

$$x(t) = A \sin(\omega_0 t)$$

where $\omega_0 = 2\pi f_0$ and A is the max. amplitude. Since $v(t) = \frac{dx}{dt}$, we have

$$v(t) = \omega_0 A \cos(\omega t)$$
$$= V_{12} \cos(\omega t)$$

$\rightarrow$ V_{12} is the max. velocity, which must just be $\omega_0 A$

$$A = \frac{V_{12}}{\omega_0} = \frac{V_{12}}{2\pi f_0} = \frac{1}{2\pi} \frac{4.76 \text{ m/s}}{5.3 \text{ Hz}} = 0.14 \text{ m}$$

d) $$T = \frac{1}{f_0} = \frac{1}{5.3 \text{ Hz}} = 0.19 \text{ s}$$

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e) Putting all the pieces together, we have:

$$x(t) = A\sin(\omega_0 t) = 0.14 \sin(33t) \quad m$$

$$v(t) = V_{12}\cos(\omega_0 t) = 4.7\cos(33t) \quad m/s$$

$\therefore$

[3] Note the $R_{eq} = 5 + 15 = 20\ \Omega$ (resistors in series)

a) Instantaneous current (i_R) through entire circuit will be:

$$i_R = \frac{V_R}{R_{eq}} = \frac{V_0\cos(\omega t)}{20\ \Omega} = 5\cos(377t)\ A$$

since $2\pi(60\,Hz) = 377\ rad/s = \omega$

<u>NOTE</u>: $f = 60\,Hz$, but $\omega = 2\pi f$ (be careful about 'frequency' versus 'angular freq.')

Peak current (I_R) will be 5.0 A (current will never be larger). So similar to instantaneous current and voltage, we can write Ohm's Law in terms of the 'peak' values:

$$V_R = I_R R = \begin{cases} 25\ V & \text{for } 5\ \Omega \text{ resistor} \\ 75\ V & \text{for } 15\ \Omega \text{ resistor} \end{cases}$$

b) At $t = 0.02\,s$, the instantaneous current is:

$$i_R(0.02) = 5\cos[2\pi(60)(0.02)] = 1.55\ A$$

By Ohm's law, $V_R = i_R R = \begin{cases} 7.7\ V & (5\ \Omega \text{ resistor}) \\ 23.2\ V & (15\ \Omega \text{ resistor}) \end{cases}$

→ Similarly note that $V_R(0.02) = 100\cos(377 \cdot 0.02) = 30.9\ V = 7.7 + 23.2$ $\therefore$

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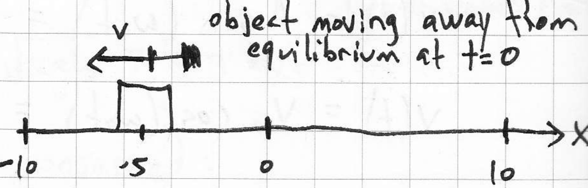
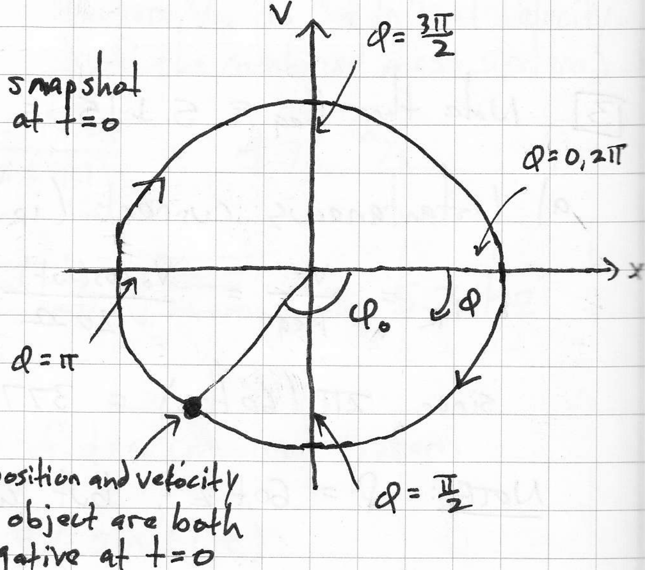
[4] Draw a picture!

- A simple picture can help get your bearings straight.
- Then drawing a picture in 'phase space' will help further! (see 3/13 notes)

Smapshot at $t=0$





Let $x(t) = A\cos(\phi)$

where $A = 10$ cm and $\phi = \phi(t) = \omega t + \phi_0$

<u>NOTE</u>: ϕ goes clockwise for increasing t

So the initial condition tells us $x(0) = -5 = 10\cos(0 + \phi_0)$

$$\text{so } \cos(\phi_0) = -\frac{1}{2} \rightarrow \phi_0 = \cos^{-1}\left(-\frac{1}{2}\right) = \pm\frac{2}{3}\pi$$

But $\phi_0 < \pi$ (reason out why from above figure!), so $\phi_0 = \frac{2}{3}\pi$. Now we have all the pieces we need:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.8_s} = 7.85 \ {}^{rad}/_{s}$$

$$\rightarrow x(t) = A\cos(\omega t + \phi_0) = 10\cos\left(7.85t + \frac{2}{3}\pi\right) \text{ cm}$$

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a) $x(2) = 10\cos\left[(7.85)(2) + \frac{2}{3}\pi\right] = 4.98$ ~~cm~~ cm

But which way is it moving? Left or right? Simplest is to determine the velocity directly:

$$v(t) = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi_0)$$

$$\rightarrow v(2) = -(7.85)(10)\sin\left[(7.85)(2) + \frac{2}{3}\pi\right] = +68.3 \text{ cm/s}$$

So the object is moving to the right since the velocity is positive

Alternatively, note that $\phi(2) = (7.85)(2) + \frac{2}{3}\pi = 17.8$ rad

$$= 4\pi + 1.67\pi \text{ rad}$$

so the 4π indicates two complete rotations and 1.67π. Looking back at our phase space plot on the last page, $\phi = 1.67\pi$ means the object is in the upper right quadrant (i.e. $x > 0$, $v > 0$)

b) $x\left(\frac{\pi}{5}\right) = 10\cos\left[(7.85)\left(\frac{\pi}{5}\right) + \frac{2\pi}{3}\right] = 7.36$ cm

$v\left(\frac{\pi}{5}\right) = -7.85 \cdot 10 \sin\left[\cdots\right] = -53.1$ cm/s

$\rightarrow$ so object is moving to the left, consistent w/

$\phi\left(\frac{\pi}{5}\right) = 7.03$ rad $\simeq 2\pi + \frac{\pi}{4}$ rads (i.e. in lower left quadrant)

$\therefore$