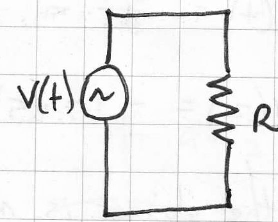
3/20/13

AC Circuits: Capacitors + Inductors

1) **Phasors** – A useful concept (introduced in Giordano ch. 22.2) for describing AC circuits



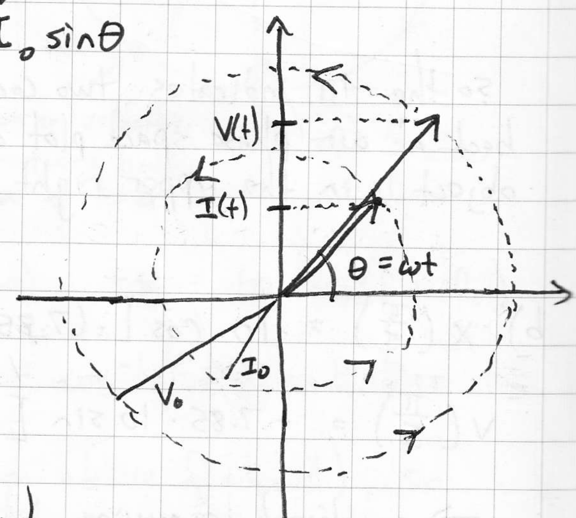
- Consider an AC voltage source connected to a resistor:

$$V(t) = V_0 \sin(\omega t) = V_0 \sin\theta$$

(i.e. $\theta = \omega t$)

- Ohm's Law still applies here:

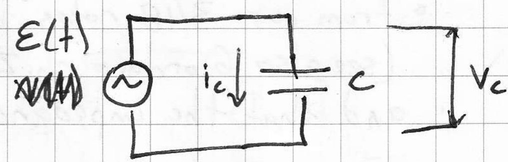
$$I(t) = \frac{V(t)}{R} = \frac{V_0}{R}\sin\theta = \underbrace{I_0}_{\text{max. current}}\sin\theta$$



- A phasor diagram is just a way to plot the dynamics via a circle
  - as time goes on, phasor rotate in counter-clockwise direction
  - phasor tail fixed at origin, head ends on a circle whose radius is max. voltage or current
  - The voltage or current can be determined at any given time (i.e. for a given value of θ) by projecting onto the vertical axis [if using cos, it'd be the horizontal axis projection]
  - The actual axes don't matter in that they do not tell you the size of I and V, but only the timing difference between them

Ac circuit w/ a capacitor

- $\Delta V_c = V_c(t) = \frac{q}{c} = \mathcal{E}(t)$



So the potential diff. across the capacitor is determined by the AC source $\mathcal{E}(t)$, which in turn determines $q = q(t)$ [C is const.!]

$\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$ — Ac voltage drive

charge on plates varies sinusoidally!

- so $q(t) = C\mathcal{E}_0 \cos(\omega t)$, but the instantaneous current through the capacitor is given by:

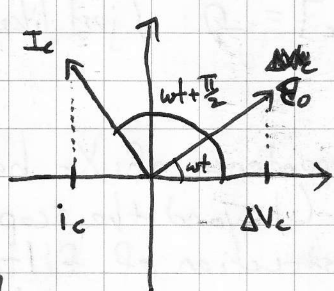
$$i_c = \frac{dq}{dt} = -\omega C\mathcal{E}_0 \sin(\omega t) = \omega C\mathcal{E}_0 \cos\left(\omega t + \frac{\pi}{2}\right)$$

phase change of 1/4 cycle!

- Note that

$v_c(t) \propto \cos(\omega t)$

$i_c(t) \propto \cos\left(\omega t + \frac{\pi}{2}\right)$



remember the projections are onto the horizontal axis here since we are using cos

→ we say that the AC current through a capacitor LEADS the voltage by 1/4 cycle

→ this is directly analogous to a simple harmonic oscillator:

$$x(t) = A\cos \omega t \;,\quad v(t) = \frac{dx}{dt} = -\omega A \sin \omega t$$

velocity

$$= -V_{max} \sin \omega t = V_{max} \cos\left(\omega t + \frac{\pi}{2}\right)$$

That is, the velocity 'leads' the position by $\frac{1}{4}$ cycle (see also Giordano Figs. 22.10 and 22.13a)

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- From our 3/18 notes, we introduced the notion of **impedance** (see also Giordano ch. 22.7) in the context of Ohm's Law and that the impedance consisted of two parts:

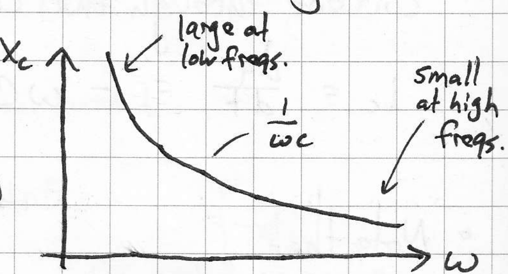
  - Resistance (describes how energy is added/removed from system)

  - Reactance (describes how energy is <u>stored</u> by the system)

→ Since a capacitor stores energy (just like a spring!), we can define its reactance

$$X_c \equiv \frac{1}{\omega C} = \frac{1}{2\pi f}$$

Note that $[X_c] = \Omega$ (just like resist.)



- This freq. dependence of X_c has many important practical implication (beyond the scope of 410 unfortunately) such for the construction of **filters** via RC circuits (see Giordano ex. 22.8 and Table 22.3) stemming from the RC time constant ($\tau = RC$) and oscillating nature of the driving voltage

- Note that for a circuit containing only a capacitor and AC voltage source, Ohm's Law still applies (see 3/18 notes!)

$$V_c = I_c X_c \leftarrow \text{reactance}$$

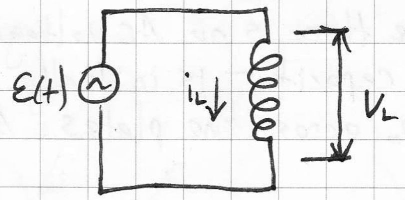
↑ max. Voltage across capacitor

↑ max. current through resistor ($= \omega C V$)

<u>Aside</u>: Using complex #'s, we actually have $X_c = -\frac{i}{\omega C}$ such that $V = -i \cdot \frac{I_c}{\omega C}$ and multiplying by $-i$ is consistent w/ V lagging I_c by $\frac{1}{4}$ cycle!

AC circuit w/ an inductor

- As we've seen before, it instantaneous voltage across an inductor is



$$V_L = L \frac{di_L}{dt}$$

where i_L is the inst. current and V_L decreases when $\frac{di_L}{dt} > 0$ and increases when $\frac{di_L}{dt} < 0$ (Lenz's law)

- If $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$, then similar to the capacitor, we'll have:

$$V_L = \mathcal{E}_0 \cos \omega t$$

(i.e. the max. volt. drops across the induct. is determined by the source)

- since $\frac{di_L}{dt} = \frac{V_L}{L} = \frac{\mathcal{E}_0}{L} \cos \omega t$, we can integrate to obtain i_L:

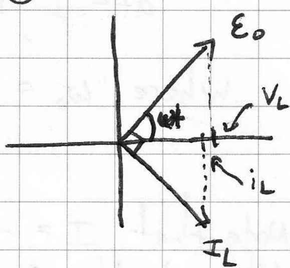
$$i_L(t) = \frac{1}{\omega} \frac{\mathcal{E}_0}{L} \sin \omega t = \frac{\mathcal{E}_0}{\omega L} \cos\left(\omega t - \frac{\pi}{2}\right) = I_L \cos\left(\omega t - \frac{\pi}{2}\right)$$

max. current thru inductor ($= \mathcal{E}_0/\omega L$)

So in this case the current lags the voltage by 1/4 cycle (see Giordano figs. 22.14 and 22.16a)

- Similar to the capacitor, the inductor stores energy and thereby has an associated reactance:

$$X_L = \omega L$$



<u>Aside</u>: When both an inductor and capacitor are present, the total reactance is

$$X = X_L - X_C = \omega L - \frac{1}{\omega C}$$

when a resistor is also in place, the total impedance is

$$Z = R + Xi = R + i\left(\omega L - \frac{1}{\omega C}\right)$$