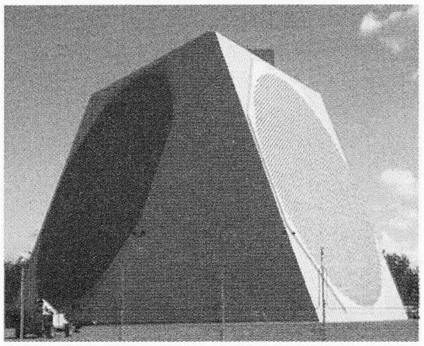
It's true that Maxwell's equations are mathematically more complex than Newton's laws and that their solution, for many problems of practical interest, requires advanced mathematics. Fortunately, we have the mathematical tools to get just far enough into Maxwell's equations to discover their most startling and revolutionary implication—the prediction of electromagnetic waves.

34.5 Electromagnetic Waves



Large radar installations like this one are used to track rockets and missiles.

It had been known since the early 19th century, from experiments on interference and diffraction, that light is a wave. We studied the wave properties of light in Part V, but at that time we were not able to determine just what is "waving."

Faraday speculated that light was somehow connected with electricity and magnetism, but Maxwell, using his equations of the electromagnetic field, was the first to understand that light is an oscillation of the electromagnetic field. Maxwell was able to predict that

- Electromagnetic waves can exist at any frequency, not just at the frequencies of visible light. This prediction was the harbinger of radio waves.
- All electromagnetic waves travel in a vacuum with the same speed, a speed that we now call the *speed of light.*

A general wave equation can be derived from Maxwell's equations, but the necessary mathematical techniques are beyond the level of this textbook. We'll adopt a simpler approach in which we *assume* an electromagnetic wave of a certain form and then show that it's consistent with Maxwell's equations. After all, the wave can't exist *unless* it's consistent with Maxwell's equations.

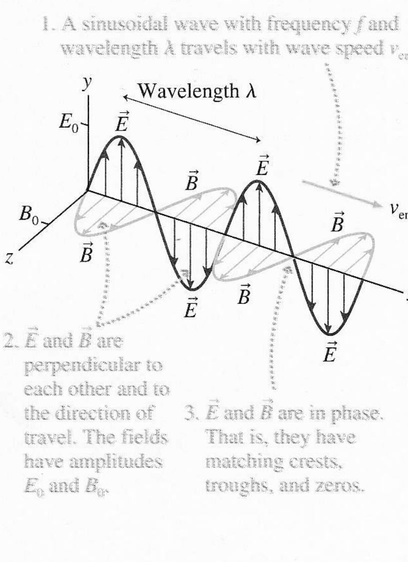
To begin, we're going to assume that electric and magnetic fields can exist independently of charges and currents in a *source-free* region of space. This is a very important assumption because it makes the statement that **fields are real entities.** They're not just cute pictures that tell us about charges and currents, but real things that can exist all by themselves. Our assertion is that the fields can exist in a self-sustaining mode in which a changing magnetic field creates an electric field (Faraday's law) that in turn changes in just the right way to re-create the original magnetic field (the Ampère-Maxwell law).

The source-free Maxwell's equations, with no charges or currents, are

$$\oint \vec{E} \cdot d\vec{A} = 0 \qquad \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_m}{dt}$$
$$\oint \vec{B} \cdot d\vec{A} = 0 \qquad \oint \vec{B} \cdot d\vec{s} = \epsilon_0 \mu_0 \frac{d\Phi_e}{dt} \tag{34.22}$$

Any electromagnetic wave traveling in empty space must be consistent with these equations.

FIGURE 34.19 A sinusoidal electromagnetic wave.



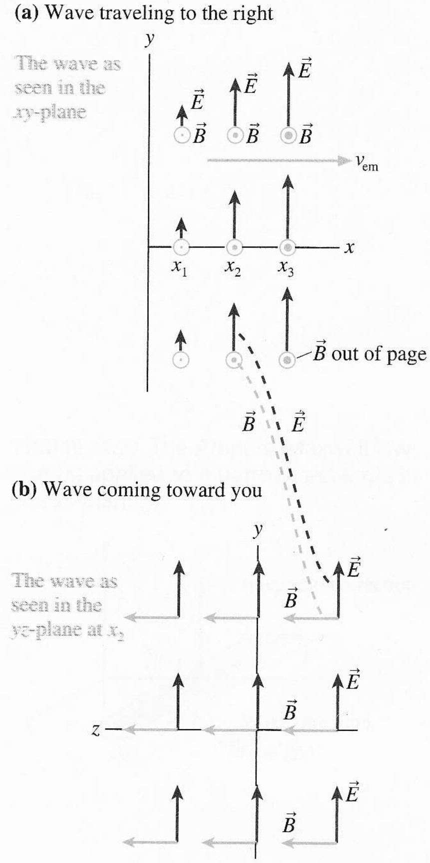
Let's postulate that an electromagnetic plane wave traveling with speed v_{em} has the characteristics shown in FIGURE 34.19. It's a useful picture, and one that you'll see in any textbook, but a picture that can be very misleading if you don't think about it carefully. $\vec{E}$ and $\vec{B}$ are *not* spatial vectors. That is, they don't stretch spatially in the y- or z-direction for a certain distance. Instead, these vectors are showing the values of the electric and magnetic fields along a single line, the x-axis. An $\vec{E}$ vector pointing in the y-direction says that *at this position* on the x-axis, where the vector's tail is, the electric field points in the y-direction and has a certain strength. Nothing is "reaching" to a point in space above the x-axis. In fact, this picture contains no information about the fields anywhere other than right on the x-axis.

However, we are assuming that this is a *plane wave,* which, you'll recall from Chapter 20, is a wave for which the fields are the same *everywhere* in any yz-plane, perpendicular to the x-axis. FIGURE 34.20a shows a small section of the xy-plane where, at this instant of time

$\vec{E}$ is pointing up and $\vec{B}$ is pointing toward you. The field strengths vary with x, the direction of travel, but not with y. As the wave moves forward, the fields that are now in the x_1-plane will soon arrive in the x_2-plane, and those now in the x_2-plane will move to x_3.

FIGURE 34.20b shows a section of the yz-plane that slices the x-axis at x_2. These fields are moving out of the page, coming toward you. The fields are the same everywhere in this plane, which is what we mean by a plane wave. If you watched a movie of the event, you would see the $\vec{E}$ and $\vec{B}$ fields at each point in this plane *oscillating* in time, but always synchronized with all the other points in the plane.

FIGURE 34.20 Interpreting the electromagnetic wave of Figure 34.19.



Gauss's Laws

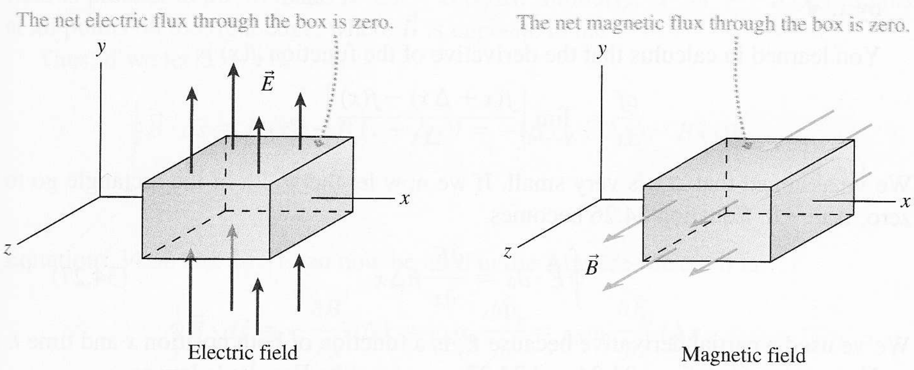
Now that we understand the shape of the electromagnetic field, we can check its consistency with Maxwell's equations. This field is a sinusoidal wave, so the components of the fields are

$$
\begin{aligned}
E_x &= 0 \quad E_y = E_0 \sin\big(2\pi(x/\lambda - ft)\big) \quad E_z = 0 \\
B_x &= 0 \quad B_y = 0 \qquad\qquad\qquad\qquad B_z = B_0 \sin\big(2\pi(x/\lambda - ft)\big)
\end{aligned}
\tag{34.23}
$$

where E_0 and B_0 are the amplitudes of the oscillating electric and magnetic fields.

FIGURE 34.21 shows an imaginary box—a Gaussian surface—centered on the x-axis. Both electric and magnetic field vectors exist at each point in space, but the figure shows them separately for clarity. $\vec{E}$ oscillates along the y-axis, so all electric field lines enter and leave the box through the top and bottom surfaces; no electric field lines pass through the sides of the box.

FIGURE 34.21 A closed surface can be used to check Gauss's law for the electric and magnetic fields.



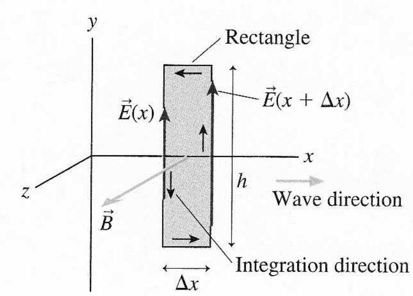
Because this is a plane wave, the magnitude of each electric field vector entering the bottom of the box is exactly matched by the electric field vector leaving the top. The electric flux through the top of the box is equal in magnitude but opposite in sign to the flux through the bottom, and the flux through the sides is zero. Thus the *net* electric flux is $\Phi_e = 0$. There is no charge inside the box because there are no sources in this region of space, so we also have $Q_{in} = 0$. Hence the electric field of a plane wave is consistent with the first of the source-free Maxwell's equations, Gauss's law.

The exact same argument applies to the magnetic field. The net magnetic flux is $\Phi_m = 0$; thus the magnetic field is consistent with the second of Maxwell's equations.

Faraday's Law

Faraday's law is concerned with the changing magnetic flux through a closed curve. We'll apply Faraday's law to a narrow rectangle in the xy-plane, shown in FIGURE 34.22, with height h and width Δx. We'll assume Δx to be so small that $\vec{B}$ is essentially constant over the width of the rectangle.

FIGURE 34.22 Faraday's law can be applied to a narrow rectangle in the xy-plane.



The magnetic field $\vec{B}$ points in the z-direction, perpendicular to the rectangle. The magnetic flux through the rectangle is $\Phi_m = B_z A_{rectangle} = B_z h\Delta x$, hence the flux *changes* at the rate

$$\frac{d\Phi_m}{dt} = \frac{d}{dt}(B_z h\Delta x) = \frac{\partial B_z}{\partial t} h\Delta x \tag{34.24}$$

The ordinary derivative dB_z/dt, which is the full rate of change of B from all possible causes, becomes a partial derivative $\partial B_z/\partial t$ in this situation because the change in magnetic flux is due entirely to the change of B with time and not at all to the spatial variation of B.

According to our sign convention, we have to go around the rectangle in a ccw direction to make the flux positive. Thus we must also use a ccw direction to evaluate the line integral

$$\oint \vec{E}\cdot d\vec{s} = \int_{\text{right}} \vec{E}\cdot d\vec{s} + \int_{\text{top}} \vec{E}\cdot d\vec{s} + \int_{\text{left}} \vec{E}\cdot d\vec{s} + \int_{\text{bottom}} \vec{E}\cdot d\vec{s} \tag{34.25}$$

The electric field $\vec{E}$ points in the y-direction, hence $\vec{E}\cdot d\vec{s} = 0$ at all points on the top and bottom edges, and these two integrals are zero.

Along the left edge of the loop, at position x, $\vec{E}$ has the same value at every point. Figure 34.22 shows that the direction of $\vec{E}$ is *opposite* to $d\vec{s}$, thus $\vec{E}\cdot d\vec{s} = -E_y(x)\,ds$. On the right edge of the loop, at position $x + \Delta x$, $\vec{E}$ is *parallel* to $d\vec{s}$ and $\vec{E}\cdot d\vec{s} = E_y(x + \Delta x)\,ds$. Thus the line integral of $\vec{E}\cdot d\vec{s}$ around the rectangle is

$$\oint \vec{E}\cdot d\vec{s} = -E_y(x)h + E_y(x + \Delta x)h = [E_y(x + \Delta x) - E_y(x)]h \tag{34.26}$$

NOTE ▶ $E_y(x)$ indicates that E_y is a function of the position x. It is *not* E_y multiplied by x. ◀

You learned in calculus that the derivative of the function $f(x)$ is

$$\frac{df}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

We've assumed that Δx is very small. If we now let the width of the rectangle go to zero, $\Delta x \to 0$, Equation 34.26 becomes

$$\oint \vec{E}\cdot d\vec{s} = \frac{\partial E_y}{\partial x} h\Delta x \tag{34.27}$$

We've used a partial derivative because E_y is a function of both position x and time t.

Now, using Equations 34.24 and 34.27, we can write Faraday's law as

$$\oint \vec{E}\cdot d\vec{s} = \frac{\partial E_y}{\partial x} h\Delta x = -\frac{d\Phi_m}{dt} = -\frac{\partial B_z}{\partial t} h\Delta x$$

The area $h\Delta x$ of the rectangle cancels, and we're left with

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \tag{34.28}$$

Equation 34.28, which compares the rate at which E_y varies with position to the rate at which B_z varies with time, is a *required condition* that an electromagnetic wave must satisfy to be consistent with Maxwell's equations. We can use Equations 34.23 for E_y and B_z to evaluate the partial derivatives:

$$\frac{\partial E_y}{\partial x} = \frac{2\pi E_0}{\lambda}\cos\big(2\pi(x/\lambda - ft)\big)$$

$$\frac{\partial B_z}{\partial t} = -2\pi f B_0 \cos\big(2\pi(x/\lambda - ft)\big)$$

Thus the required condition of Equation 34.28 is

$$\frac{\partial E_y}{\partial x} = \frac{2\pi E_0}{\lambda}\cos\big(2\pi(x/\lambda - ft)\big) = -\frac{\partial B_z}{\partial t} = 2\pi f B_0 \cos\big(2\pi(x/\lambda - ft)\big)$$

Canceling the many common factors, and multiplying by λ, we're left with

$$E_0 = (\lambda f)B_0 = v_{em}B_0 \tag{34.29}$$

where we used the fact that $\lambda f = v$ for any sinusoidal wave.

Equation 34.29, which came from applying Faraday's law, tells us that the field amplitudes E_0 and B_0 of an electromagnetic wave are not arbitrary. **Once the amplitude B_0 of the magnetic field wave is specified, the electric field amplitude E_0 must be $E_0 = v_{em}B_0$.** Otherwise the fields won't satisfy Maxwell's equations.

The Ampère-Maxwell Law

We have one equation to go, but this one will now be easier. The Ampère-Maxwell law is concerned with the changing electric flux through a closed curve. FIGURE 34.23 shows a very narrow rectangle of width Δx and length l in the xz-plane. The electric field is perpendicular to this rectangle; hence the electric flux through it is $\Phi_e = E_y A_{rectangle} = E_y l \Delta x$. This flux is changing at the rate

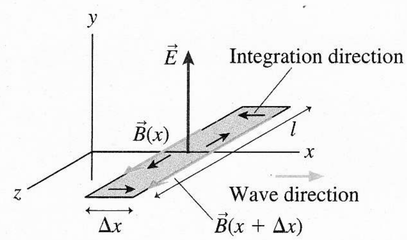


FIGURE 34.23 The Ampère-Maxwell law can be applied to a narrow rectangle in the xz-plane.

$$\frac{d\Phi_e}{dt} = \frac{d}{dt}(E_y l \Delta x) = \frac{\partial E_y}{\partial t} l \Delta x \tag{34.30}$$

The line integral of $\vec{B}\cdot d\vec{s}$ around this closed rectangle is calculated just like the line integral of $\vec{E}\cdot d\vec{s}$ in Figure 34.22. $\vec{B}$ is perpendicular to $d\vec{s}$ on the narrow ends, so $\vec{B}\cdot d\vec{s} = 0$. The field at *all* points on the left edge, at position x, is $\vec{B}(x)$, and this field is parallel to $d\vec{s}$ to make $\vec{B}\cdot d\vec{s} = B_z(x)\,ds$. Similarly, $\vec{B}\cdot d\vec{s} = -B_z(x+\Delta x)\,ds$ at all points on the right edge, where $\vec{B}$ is opposite to $d\vec{s}$.

Thus, if we let $\Delta x \to 0$,

$$\begin{aligned}\oint \vec{B}\cdot d\vec{s} &= B_z(x)l - B_z(x+\Delta x)l = -[B_z(x+\Delta x) - B_z(x)]l \\ &= -\frac{\partial B_z}{\partial x} l \Delta x\end{aligned} \tag{34.31}$$

Equations 34.30 and 34.31 can now be used in the Ampère-Maxwell law:

$$\oint \vec{B}\cdot d\vec{s} = -\frac{\partial B_z}{\partial x} l\Delta x = \epsilon_0\mu_0 \frac{d\Phi_e}{dt} = \epsilon_0\mu_0\frac{\partial E_y}{\partial t} l \Delta x$$

The area of the rectangle cancels, and we're left with

$$\frac{\partial B_z}{\partial x} = -\epsilon_0\mu_0 \frac{\partial E_y}{\partial t} \tag{34.32}$$

Equation 34.32 is a second required condition that the fields must satisfy. If we again evaluate the partial derivatives, using Equations 34.23 for E_y and B_z, we find

$$\frac{\partial E_y}{\partial t} = -2\pi f E_0 \cos\big(2\pi(x/\lambda - ft)\big)$$

$$\frac{\partial B_z}{\partial x} = \frac{2\pi B_0}{\lambda}\cos\big(2\pi(x/\lambda - ft)\big)$$

With these, Equation 34.32 becomes

$$\frac{\partial B_z}{\partial x} = \frac{2\pi B_0}{\lambda}\cos\big(2\pi(x/\lambda - ft)\big) = -\epsilon_0\mu_0\frac{\partial E_y}{\partial t} = 2\pi\epsilon_0\mu_0 f E_0 \cos\big(2\pi(x/\lambda - ft)\big)$$

A final round of cancellations and another use of $\lambda f = v_{em}$ leave us with

$$E_0 = \frac{B_0}{\epsilon_0 \mu_0 \lambda f} = \frac{B_0}{\epsilon_0 \mu_0 v_{em}} \tag{34.33}$$

The last of Maxwell's equations gives us another constraint between E_0 and B_0.

The Speed of Light

But how can Equation 34.29, which required $E_0 = v_{em} B_0$, and Equation 34.33 both be true at the same time? The one and only way is if

$$\frac{1}{\epsilon_0 \mu_0 v_{em}} = v_{em}$$

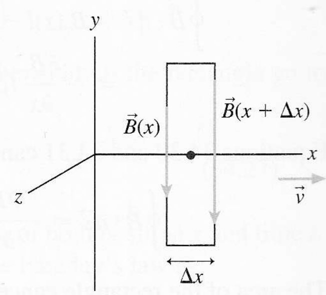
from which we find

$$v_{em} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s} = c \tag{34.34}$$

This is a remarkable conclusion. The constants ϵ_0 and μ_0 are from electrostatics and magnetostatics, where they determine the size of $\vec{E}$ and $\vec{B}$ due to point charges. Coulomb's law and the Biot-Savart law, where ϵ_0 and μ_0 first appeared, have nothing to do with waves. Yet Maxwell's theory of electromagnetism ends up predicting that electric and magnetic fields can form a self-sustaining electromagnetic wave *if* that wave travels at the specific speed $v_{em} = 1/\sqrt{\epsilon_0 \mu_0}$. No other speed will satisfy Maxwell's equations.

We've made no assumption about the frequency of the wave, so apparently all electromagnetic waves, regardless of their frequency, travel (in vacuum) at the same speed $v_{em} = 1/\sqrt{\epsilon_0 \mu_0}$. We call this speed c, the "speed of light," but it applies equally well from low-frequency radio waves to ultrahigh-frequency x rays.

STOP TO THINK 34.3 An electromagnetic wave is propagating in the positive x-direction. At this instant of time, what is the direction of $\vec{E}$ at the center of the rectangle?



a. In the positive x-direction
b. In the negative x-direction
c. In the positive y-direction
d. In the negative y-direction
e. In the positive z-direction
f. In the negative z-direction

34.6 Properties of Electromagnetic Waves

We've demonstrated that one very specific sinusoidal wave is consistent with Maxwell's equations. It's possible to show that *any* electromagnetic wave, whether it's sinusoidal or not, must satisfy four basic conditions:

1. The fields $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation $\vec{v}_{em}$. Thus an electromagnetic wave is a transverse wave.
2. $\vec{E}$ and $\vec{B}$ are perpendicular to each other in a manner such that $\vec{E} \times \vec{B}$ is in the direction of $\vec{v}_{em}$.
3. The wave travels in vacuum at speed $v_{em} = 1/\sqrt{\epsilon_0 \mu_0} = c$.
4. $E = cB$ at any point on the wave.

In this section, we'll look at some other properties of electromagnetic waves.

Energy and Intensity

Waves transfer energy. Ocean waves erode beaches, sound waves set your eardrums vibrating, and light from the sun warms the earth. The energy flow of an electromagnetic wave is described by the **Poynting vector** $\vec{S}$, defined as

$$\vec{S} \equiv \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{34.35}$$

The Poynting vector, shown in FIGURE 34.24, has two important properties:

1. The Poynting vector points in the direction in which an electromagnetic wave is traveling. You can see this by looking back at Figure 34.19.
2. It is straightforward to show that the units of S are W/m^2, or power (joules per second) per unit area. Thus the magnitude S of the Poynting vector measures the rate of energy transfer per unit area of the wave.

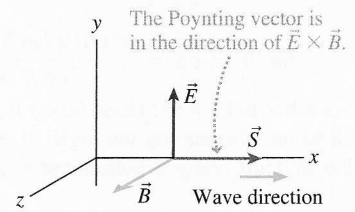


FIGURE 34.24 The Poynting vector.

Because $\vec{E}$ and $\vec{B}$ of an electromagnetic wave are perpendicular to each other, and $E = cB$, the magnitude of the Poynting vector is

$$S = \frac{EB}{\mu_0} = \frac{E^2}{c\mu_0} = c\epsilon_0 E^2$$

The Poynting vector is a function of time, oscillating from zero to $S_{max} = E_0^2/c\mu_0$ and back to zero twice during each period of the wave's oscillation. That is, the energy flow in an electromagnetic wave is not smooth. It "pulses" as the electric and magnetic fields oscillate in intensity. We're unaware of this pulsing because the electromagnetic waves that we can sense—light waves—have such high frequencies.

Of more interest is the *average* energy transfer, averaged over one cycle of oscillation, which is the wave's **intensity** I. In our earlier study of waves, we defined the intensity of a wave to be $I = P/A$, where P is the power (energy transferred per second) of a wave that impinges on area A. Because $E = E_0 \sin\big(2\pi(x/\lambda - ft)\big)$, and the average over one period of $\sin^2\big(2\pi(x/\lambda - ft)\big)$ is $\frac{1}{2}$, the intensity of an electromagnetic wave is

$$I = \frac{P}{A} = S_{avg} = \frac{1}{2c\mu_0}E_0^2 = \frac{c\epsilon_0}{2}E_0^2 \tag{34.36}$$

Equation 34.36 relates the intensity of an electromagnetic wave, a quantity that is easily measured, to the amplitude of the wave's electric field.

The intensity of a plane wave, with constant electric field amplitude E_0, would not change with distance. But a plane wave is an idealization; there are no true plane waves in nature. You learned in Chapter 20 that, to conserve energy, the intensity of a wave far from its source decreases with the inverse square of the distance. If a source with power P_{source} emits electromagnetic waves *uniformly* in all directions, the electromagnetic wave intensity at distance r from the source is

$$I = \frac{P_{source}}{4\pi r^2} \tag{34.37}$$

Equation 34.37 simply expresses the recognition that the energy of the wave is spread over a sphere of surface area $4\pi r^2$.

EXAMPLE 34.4 Fields of a cell phone

A digital cell phone broadcasts a 0.60 W signal at a frequency of 1.9 GHz. What are the amplitudes of the electric and magnetic fields at a distance of 10 cm, about the distance to the center of the user's brain?

MODEL Treat the cell phone as a point source of electromagnetic waves.

Continued

SOLVE The intensity of a 0.60 W point source at a distance of 10 cm is

$$I = \frac{P_{source}}{4\pi r^2} = \frac{0.60\ \text{W}}{4\pi(0.10\ \text{m})^2} = 4.78\ \text{W/m}^2$$

We can find the electric field amplitude from the intensity:

$$E_0 = \sqrt{\frac{2I}{c\epsilon_0}} = \sqrt{\frac{2(4.78\ \text{W/m}^2)}{(3.00 \times 10^8\ \text{m/s})(8.85 \times 10^{-12}\ \text{C}^2/\text{N m}^2)}}$$

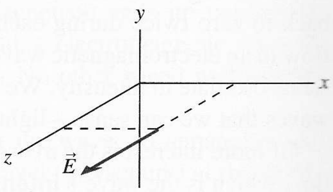
$$= 60\ \text{V/m}$$

The amplitudes of the electric and magnetic fields are related by the speed of light. This allows us to compute

$$B_0 = \frac{E_0}{c} = 2.0 \times 10^{-7}\ \text{T}$$

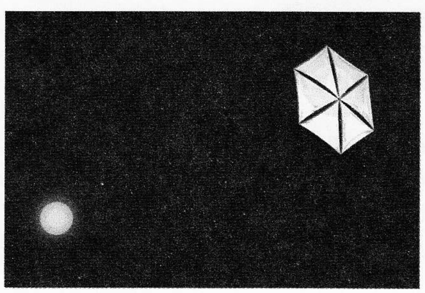
ASSESS The electric field amplitude is modest; the magnetic field amplitude is very small. This implies that the interaction of electromagnetic waves with matter is mostly due to the electric field.

STOP TO THINK 34.4 An electromagnetic wave is traveling in the positive y-direction. The electric field at one instant of time is shown at one position. The magnetic field at this position points



a. In the positive x-direction.
b. In the negative x-direction.
c. In the positive y-direction.
d. In the negative y-direction.
e. Toward the origin.
f. Away from the origin.

Radiation Pressure



Artist's conception of a future spacecraft powered by radiation pressure from the sun.

Electromagnetic waves transfer not only energy but also momentum. An object gains momentum when it absorbs electromagnetic waves, much as a ball at rest gains momentum when struck by a ball in motion.

Suppose we shine a beam of light on an object that completely absorbs the light energy. If the object absorbs energy during a time interval Δt, its momentum changes by

$$\Delta p = \frac{\text{energy absorbed}}{c}$$

This is a consequence of Maxwell's theory, which we'll state without proof.

The momentum change implies that the light is exerting a force on the object. Newton's second law, in terms of momentum, is $F = \Delta p/\Delta t$. The radiation force due to the beam of light is

$$F = \frac{\Delta p}{\Delta t} = \frac{(\text{energy absorbed})/\Delta t}{c} = \frac{P}{c}$$

where P is the power (joules per second) of the light.

It's more interesting to consider the force exerted on an object per unit area, which is called the **radiation pressure** p_{rad}. The radiation pressure on an object that absorbs all the light is

$$p_{rad} = \frac{F}{A} = \frac{P/A}{c} = \frac{I}{c} \tag{34.38}$$

where I is the intensity of the light wave. The subscript on p_{rad} is important in this context to distinguish the radiation pressure from the momentum p.