

Angular momentum \vec{L}

①

(i) \vec{L} of a pointlike particle defined by
$$\vec{L} = \vec{r} \times \vec{p}$$

e.g. for the ball attached to massless rod discussed above we have:

$$\begin{aligned} |\vec{L}| &= |\vec{r}| |\vec{p}| \sin \phi \\ &= r m v \quad (\phi = \frac{\pi}{2} \text{ for circular motion}) \\ &= mr^2 \omega = I \omega \end{aligned}$$

orientation: RH rule $\subset \circ$ $\vec{L} = I \vec{\omega}$

Note that for uniform circular motion $\vec{\omega} = \text{const} \Rightarrow \vec{L} = \text{constant}$.

Now recall Newton-2: $\frac{d}{dt} \vec{p} = \vec{F}$
(see discussion of impulse theorem)

Try something similar with \vec{L} : product rule

$$\begin{aligned} \Leftarrow \frac{d}{dt} \vec{L} &= \frac{d}{dt} (\vec{r} \times \vec{p}) \stackrel{\downarrow}{=} \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \\ &= m \underbrace{\vec{v} \times \vec{v}}_{=0} + \vec{r} \times \vec{F} = \vec{\tau} \end{aligned}$$

Summary: $\boxed{\frac{d}{dt} \vec{L} = \vec{\tau}}$ angular-momentum theorem

We can infer a conservation law:

$$\begin{aligned} &\text{if } \vec{\tau} = 0 \Rightarrow \vec{L} \text{ conserved} \\ \vec{\tau} = 0 &\text{ if } \begin{cases} \vec{F} = 0 & (\text{trivial}) \\ \vec{r} \parallel \vec{F} & (\text{e.g. uniform circular motion}) \end{cases} \end{aligned}$$

(ii) \vec{L} for a particle system (extended object)

define

$$\vec{L}_{tot} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (m_i \vec{r}_i \times \vec{v}_i)$$

$$\vec{\tau}_{tot} = \sum_i \vec{r}_i \times \vec{F}_i$$

show (similarly as above): $\frac{d}{dt} \vec{L}_{tot} = \vec{\tau}_{tot}$ *)

For rotations about a fixed axis, one finds (again):

$$\vec{L}_{tot} = I \vec{\omega} \quad \text{with} \quad I = \sum_i m_i r_i^2$$

Examples for angular momentum conservation

(i) Spinning figure skater

$$L_{initial} \stackrel{\vec{v}=0}{=} I_{initial} \omega_{initial} = L_{final} = I_{final} \omega_{final} \quad \Leftrightarrow \quad \omega_{final} = \frac{I_{initial}}{I_{final}} \omega_{initial}$$

start with a small angular velocity $\omega_{initial}$, then pull arms and legs in to reduce $I \Rightarrow$ end up with a higher angular velocity ω_{final}

Consider KE:

$$KE_f = \frac{1}{2} I_{final} \omega_{final}^2 = \frac{1}{2} I_{final} \left(\frac{I_{initial}}{I_{final}} \omega_{initial} \right)^2$$

$$= \left(\frac{I_{initial}}{I_{final}} \right) \frac{1}{2} I_{initial} \omega_{initial}^2 = \frac{I_{initial}}{I_{final}} KE_i > 1$$

$$\therefore KE_f > KE_i$$

KE is not conserved, but increased (the skater does positive work when pulling in arms+legs)

2) Angular momentum conservation, if $\vec{\tau}_{tot} = 0$:

$$\vec{L}_{tot} = \vec{L}_i = \vec{L}_f = \text{const}$$

(ii) Planetary motion

Recall : Earth orbits Sun on an elliptical orbit \leftarrow)

$$\vec{F}_{SunE} = - \frac{GM_E M_S}{r^3} \vec{r}$$

Earth's position vector relative to resting sun

Q : does Sun exert a torque on Earth ?

$$\vec{\tau} = \vec{r} \times \vec{F}_{SunE} = - \frac{GM_E M_S}{r^3} \underbrace{\vec{r} \times \vec{r}}_{=0} = 0$$

A : no $\Rightarrow \vec{L}_{Earth} = \text{const}$

This has two important implications :

(i) Direction of \vec{L}_{Earth} never changes \Rightarrow orbit is truly planar

(ii) Magnitude $|\vec{L}_{Earth}|$ never changes ;
one can show that Kepler's area law follows from this fact

(see book, Chap 9.4)

So, Sun doesn't exert a torque on Earth. It does do work though. For Earth's (slightly) elliptical orbit \vec{F}_{grav} is not always perpendicular to displacement (unlike for uniform circular motion).

*) Assume Sun is infinitely heavy, i.e. $M_S \rightarrow \infty$ (such that it is at rest despite Newton-3.)