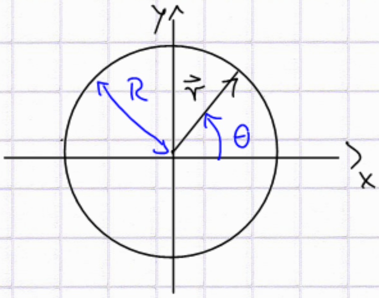
Circular motion

(i) Uniform circular motion



characteristics.

$R = \text{const}$ (radius)

$v = \text{const}$ (speed)

$T = \frac{2\pi R}{v} = \text{const}$ (period)

Alternatively, define frequency $f = \frac{1}{T}$

and angular frequency $\omega = 2\pi f = \frac{2\pi}{T} = \frac{v}{R}$

Goal: complete kinematical description of (uniform) circular motion

- circ. trajectory:

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$
$$= R\cos\theta(t)\,\hat{i} + R\sin\theta(t)\,\hat{j}$$

check: $|\vec{r}(t)| = \sqrt{x^2(t) + y^2(t)}$

$$= \sqrt{R^2\left(\cos^2\theta(t) + \sin^2\theta(t)\right)} = R$$

- velocity vector (use chain rule → math addendum)

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = -R\dot{\theta}\sin\theta\,\hat{i} + R\dot{\theta}\cos\theta\,\hat{j}$$

speed: $v = |\vec{v}(t)| = \sqrt{R^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta)} = R\dot{\theta}$

compare to previous eq.: $v = \omega R$

and identify $\omega = \dot{\theta} = \text{const}$ (circular frequency = angular velocity)

$\Rightarrow \theta(t) = \omega t \; (+\theta_0)$ (uniform circ. motion)

- acceleration (use $\theta = \omega t$ from now on):

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = -R\omega \frac{d}{dt}\left(\sin(\omega t)\hat{\imath}\right) + R\omega \frac{d}{dt}\left(\cos(\omega t)\hat{\jmath}\right)$$

$$= -\omega^2 R\cos(\omega t)\hat{\imath} - \omega^2 R \sin(\omega t)\hat{\jmath}$$

compare with $\vec{r}(t)$

$$= -\omega^2 \vec{r}(t)$$

- $\vec{a}(t)$ points toward centre of circle (opposite to $\vec{r}(t)$)
- $a_c := |\vec{a}(t)| = \omega^2 |\vec{r}(t)| = \omega^2 R = \text{const}$

  "centripetal" (centre seeking) acceleration

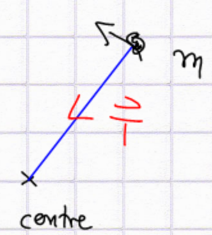
  note that due to $\omega = \frac{v}{R}$ one can also write $a_c = \frac{v^2}{R}$

We are done with the kinematics, but:
What about the dynamics of uniform circular motion?

→ Newton-2 says: $m\vec{a} = \vec{F}_{net}$

→ need $F_{net} = m a_c = \frac{mv^2}{R}$ (with $\vec{F}_{net}$ pointing toward centre of circle) to obtain uniform circular motion.

ex ①: a rock tied to a string being hurled around (in interstellar space, i.e. w/o gravity)

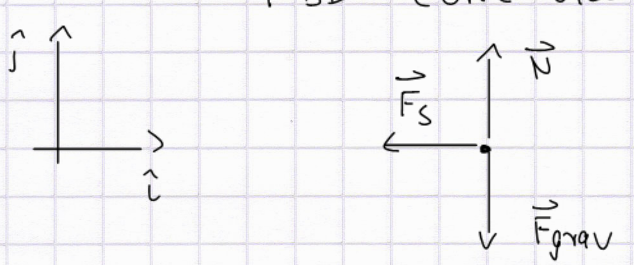


→ tension $\vec{T}$ is the only acting force

hence: $m a_c = \frac{mv^2}{R} = |\vec{T}|$

ex ②: a turning car (see book, Fig 5.6)

FBD (end view)



you need (static) friction to make the turn:

$$F_{net} = F_s = m a_c = \frac{mv^2}{R} \leq \mu_s N$$

radius of curvature

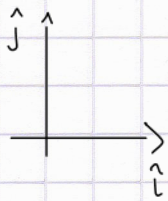
maximum speed: (for given R, μ_s)

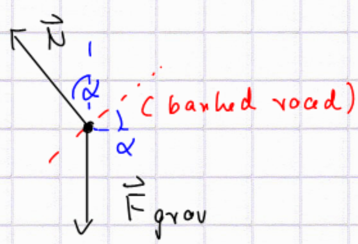
$$\frac{m v_{max}^2}{R} = \mu_s N = \mu_s m g$$

$$\Longleftrightarrow \quad v_{max} = \sqrt{\mu_s g R}$$

if $v > v_{max}$, car slips! If you want to go faster, you need to bank the turn. Then you can even make it trough the curve on perfect ice (i.e. w/o friction):

FBD (end view)





condition:

$$\vec{N} + \vec{F}_{grav} = -\frac{m v^2}{R}\hat{\imath}$$

(i.e. forces need to add up to a purely centripetal force)

$\hat{j}$: $\quad N \cos\alpha = F_{grav} \quad \Longleftrightarrow \quad N = \dfrac{mg}{\cos\alpha}$

$\hat{\imath}$: $\quad N \sin\alpha = mg\tan\alpha = \dfrac{m v^2}{R} \quad \Longleftrightarrow \quad \boxed{v^2 = g\tan\alpha}$

(ii) Nonuniform circular motion

start over from:

$$\vec{r}(t) = R\cos\theta(t)\,\hat{\imath} + R\sin\theta(t)\,\hat{j}$$

but this time $\dot{\theta}$ is not constant.

compare with linear motion to proceed; there, we considered

constant linear acceleration: $\quad x(t) = x_0 + v_0 t + \frac{a}{2}t^2 \quad$ (1D)

so "ansatz": $\quad \theta(t) = \theta_0 + \omega_0 t + \frac{\alpha}{2}t^2 \quad$ (yet undetermined)

play around: $\quad \omega = \dot{\theta} = \omega_0 + \alpha t$

$\qquad\qquad\qquad \dot{\omega} = \ddot{\theta} = \alpha = \text{const} \qquad$ angular acceleration i.e we are considering the case of circular motion w/ constant angular acceleration

Let's work it through:

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = -R\dot{\theta}\sin\theta\,\hat{i} + R\dot{\theta}\cos\theta\,\hat{j}$$

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = -R\frac{d}{dt}\left(\dot{\theta}\sin\theta\right)\hat{i} + R\frac{d}{dt}\left(\dot{\theta}\cos\theta\right)\hat{j}$$

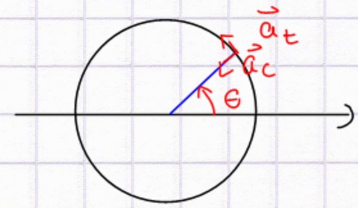
$$= -R\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\hat{i} + R\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\hat{j}$$

$$= -R\left(\alpha\sin\theta + \omega^2\cos\theta\right)\hat{i} + R\left(\alpha\cos\theta - \omega^2\sin\theta\right)\hat{j}$$

$$\overset{\text{rearrange}}{=} -\omega^2\left(R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j}\right) - R\alpha\left(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}\right)$$

$$\overset{\text{compare}}{=} \underbrace{-\omega^2\vec{r}(t)}_{\parallel} + \underbrace{\frac{\alpha}{\omega}\vec{v}(t)}_{\parallel}$$

$$=: \vec{a}_c(t) + \vec{a}_t(t)$$



Comments

- One can prove that $\vec{a}_c(t) \perp \vec{a}_t(t)$ (tutorial)
- $\vec{a}(t)$ does not point to centre of circle if $\alpha \neq 0$
- $a_c = |\vec{a}_c(t)| = \omega^2(t)\,R = \frac{v^2(t)}{R}$

  $a_t = |\vec{a}_t(t)| = \left|\frac{\alpha}{\omega}\right| v = R\alpha$ (assume counter-clockwise rotation so that $|\omega| = \omega > 0$ ...)

  relate this to changing speed:

$$\frac{d}{dt}v(t) = R\frac{d\omega}{dt} = R\alpha = a_t$$

(more math details were discussed in tutorial)