# Centre of mass (CM)



§ 2 particles

add Newton-2 : $m_1 \vec{a}_1 + m_2 \vec{a}_2 = \vec{F}_1 + \vec{F}_2$

with $\vec{F}_1$ = total force on $m_1$
$\vec{F}_2$ = total force on $m_2$

$$\vec{F}_1 + \vec{F}_2 = \sum \vec{F}_{ext} + \underbrace{\sum \vec{F}_{int}}_{= \vec{F}_{1 on 2} + \vec{F}_{2 on 1} \overset{\text{Newton-3}}{=} 0} = \sum \vec{F}_{ext}$$

hence : $m_1 \vec{a}_1 + m_2 \vec{a}_2 = \sum \vec{F}_{ext} =: M_{tot} \vec{a}_{CM}$

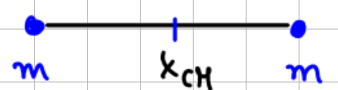
with $M_{tot} = m_1 + m_2$, $\vec{a}_{CM} = \dfrac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{M_{tot}}$

now define : $\vec{v}_{CM} = \dfrac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{M_{tot}}$ such that $\dfrac{d}{dt} \vec{v}_{CM} = \vec{a}_{CM}$

similarly : $\vec{r}_{CM} := \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M_{tot}}$ such that $\dfrac{d}{dt} \vec{r}_{CM} = \vec{v}_{CM}$

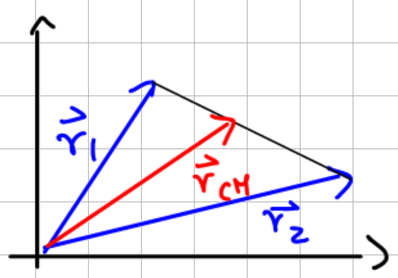
this is the "centre of mass" of a two-particle system.

special case : 1D, $m_1 = m_2 \equiv m$ : $x_{CM} = \dfrac{m x_1 + m x_2}{2m} = \dfrac{x_1 + x_2}{2}$




(midpoint!)

in 2D (and 3D)




(endpoint of $\vec{r}_{CM}$ located on line that connects $\vec{r}_1$ and $\vec{r}_2$.)

Generalization: CM of N-particle system

$$\vec{r}_{CM} := \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M_{tot}} \qquad \left( M_{tot} = \sum_{i=1}^{N} m_i \right)$$

$$\Rightarrow \vec{V}_{CM} = \frac{d}{dt}\vec{r}_{CM} = \frac{\sum_i m_i \vec{V}_i}{M_{tot}} = \frac{\sum_i \vec{p}_i}{M_{tot}} = \frac{\vec{P}_{tot}}{M_{tot}}$$

$$\Leftrightarrow \vec{P}_{tot} = M_{tot}\vec{V}_{CM}$$

(total momentum = CM momentum)

$$\Rightarrow \vec{a}_{CM} = \frac{d}{dt}\vec{V}_{CM} = \frac{\sum_i m_i \vec{a}_i}{M_{tot}}$$

$$\Leftrightarrow \boxed{M_{tot}\vec{a}_{CM} = \sum_i m_i \vec{a}_i = \sum \vec{F}_{ext}}$$

$\sum \vec{F}_{int} = 0$ due to Newton-3.

CM of a particle system moves as if system were a single particle of mass $M_{tot}$ acted on by total external force.

$\Rightarrow$ CM describes "translational motion" of particle system "as a whole".

rewrite last equation:

$$M_{tot}\vec{a}_{CM} = \frac{d}{dt}\left(M_{tot}\vec{V}_{CM}\right) = \frac{d}{dt}\vec{P}_{tot} = \sum \vec{F}_{ext}$$

For a closed system we have $\sum \vec{F}_{ext} = 0$

hence $\frac{d}{dt}\vec{P}_{tot} = 0 \Rightarrow \vec{P}_{tot}$ conserved

(i.e. we recover total momentum conservation)