Kinetic energy of rotation

Starting point:
KE of a system of masses

$$KE = \sum_i \frac{m_i}{2} v_i^2$$

→ rotation of extended object about fixed axis $\vec{\omega}$.

For each piece of mass we have $v_i = \omega r_i$

with the same ω for all m_i if object is rigid (i.e. can't be deformed).

$$\Rightarrow KE_{rot} = \sum_i \frac{m_i}{2}(\omega r_i)^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$
$$= \frac{1}{2} I \omega^2$$

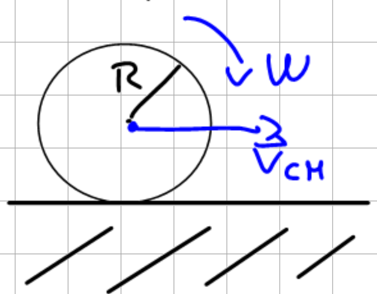
(compare this with $KE = \frac{1}{2} m v^2$ for a mass point: $m \to I$, $v \to \omega$)

An extended object can both rotate and translate.

One finds for the total kinetic energy:

$$KE_{tot} = KE_{trans} + KE_{rot}$$
$$= \frac{M}{2} v_{CM}^2 + \frac{I}{2}\omega^2 \quad \text{(if } \vec{\omega} \text{ passes through CM)}$$
$$(M = \textstyle\sum_i m_i)$$

example: rolling motion



→ one revolution (no slipping):

$$d = 2\pi R, \quad T = \frac{2\pi}{\omega}$$
$$\Rightarrow v_{CM} = \frac{d}{T} = \omega R$$

(v_{CM} locked with ω)

$\Rightarrow \quad KE_{tot} = \frac{M}{2} v_{CM}^2 + \frac{I}{2}\left(\frac{v_{CM}}{R}\right)^2$

- consider rolling <u>disk</u>: $\quad I = \frac{M}{2}R^2$

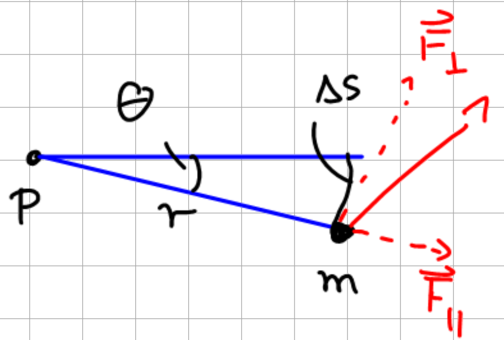
$$= \frac{M}{2} v_{CM}^2 + \frac{M}{4} v_{CM}^2 = \frac{3}{4} M v_{CM}^2$$

- consider rolling wheel: $\quad I \approx MR^2$

(assuming wheel ≈ thin hoop)

$$KE_{tot} = \frac{M}{2} v_{CM}^2 + \frac{M}{2} v_{CM}^2 = M v_{CM}^2$$

Now work for rotational motion: consider – once again – the ball attached to the massless rod:



decompose applied force:

$$\vec{F} = \vec{F}_\perp + \vec{F}_\parallel$$

$$W = \vec{F}\cdot\Delta\vec{s} = \vec{F}_\perp\cdot\Delta\vec{s} + \underbrace{\vec{F}_\parallel\cdot\Delta\vec{s}}_{=0} = F_\perp \Delta s$$

compare with $\quad \tau = r F_\perp \iff F_\perp = \frac{\tau}{r}$

$$\Rightarrow \quad W = \tau \frac{\Delta s}{r} = \tau\,\theta$$

On the other hand, work-energy theorem still holds:

$$W = \Delta KE = \frac{m}{2} v_f^2 - \frac{m}{2} v_i^2$$

$$= \frac{m}{2}(\omega_f r)^2 - \frac{m}{2}(\omega_i r)^2$$

$$= \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 = \Delta KE_{rot} \qquad (I = mr^2)$$

$\Rightarrow$ work-energy theorem for rotational motion: $\quad \boxed{W = \tau\theta = \Delta KE_{rot}}$

Finally, we can consider combined translational and rotational motion:

$$KE_{tot} = KE_{trans} + KE_{rot}$$

$$W_{tot} = \Delta KE_{tot} = \Delta KE_{trans} + \Delta KE_{rot}$$

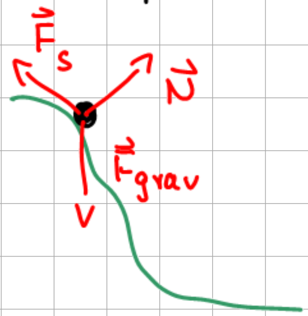
$$= PE_i - PE_f \quad \text{(if forces are conservative)}$$

$$= -\Delta PE$$

$$\Leftrightarrow \quad \boxed{\Delta KE_{trans} + \Delta KE_{rot} + \Delta PE = 0}$$

(conservation of total mechanical energy)

example: ball rolling down a hill



recall: • rolling motion $\longleftrightarrow$ static friction $\vec{F}_s$

$\hookrightarrow$ static friction doesn't do work!

• $\vec{N} \perp \Delta\vec{r} \Rightarrow \vec{N}$ doesn't do work

$$\hookrightarrow \quad \Delta PE = PE_f - PE_i = mg\underset{\overset{\shortparallel}{0}}{h_f} - mg\underset{\overset{\shortparallel}{h}}{h_i} = -mgh$$

(same as for sliding down w/o friction, i.e. rolling motion doesn't affect PE.)

$$KE_{tot} = KE_{trans} + KE_{rot} = \frac{m}{2}v_{CM}^2 + \frac{I}{2}\omega^2$$

$$KE_i = 0 \qquad \text{(assume ball starts from rest)}$$

$$\Delta KE_{tot} = KE_f = \frac{m}{2}v_{CM,f}^2 + \frac{I}{2}\omega_f^2 = \frac{m}{2}v_{CM,f}^2 + \frac{I}{2}\left(\frac{v_{CM,f}}{R}\right)^2$$

ball = solid sphere of radius R: $I = \frac{2}{5}mR^2$

(is this really true?)

$$\hookrightarrow \quad \Delta KE_{tot} + \Delta PE = \frac{m}{2}v_{CM,f}^2 + \frac{m}{5}v_{CM,f}^2 - mgh = 0$$

$$\Leftrightarrow \left(\frac{1}{2}+\frac{1}{5}\right) \cancel{m}\, v_{CM,f}^2 = \cancel{m} g h$$

$$\Leftrightarrow v_{CM,f} = \sqrt{\frac{10gh}{7}} < \underbrace{v_f = \sqrt{2gh}}_{\text{result for sliding w/o friction}}$$

The final velocity of the rolling ball is smaller than that of a (frictionless) sliding object, since some of the initial potential energy is needed for the rotational motion.