Forces and Motion

example 1: ball rolling on and falling off a table (again)

→ now math description of position–time graphs:

$x(t) = c_1 t$
with $c_1 = \text{const}$

$$y(t) = \begin{cases} h & \text{if } t \le 0 \\ h - c_2 t^2 & \text{if } t \ge 0 \end{cases}$$

$v_x = \frac{dx}{dt} = \frac{d}{dt}(c_1 t)$

$= c_1 \equiv v_{0,x}$

$$v_y = \frac{dy}{dt} = \begin{cases} 0 & \text{if } t \le 0 \\ -2c_2 t & \text{if } t \ge 0 \end{cases}$$

$a_x = \frac{dv_x}{dt} = 0$

$$a_y = \frac{dv_y}{dt} = \begin{cases} 0 & \text{if } t < 0 \\ -2c_2 = -g & \text{if } t > 0 \end{cases}$$

$\Rightarrow$ $F_{x,net} = ma_x = 0$

(note that we can't calculate a_y @ $t=0$)

$$\Rightarrow F_{y,net} = ma_y = \begin{cases} 0 & \text{if } t < 0 \\ -mg & \text{if } t > 0 \end{cases}$$

We have used a few differential calculus rules:

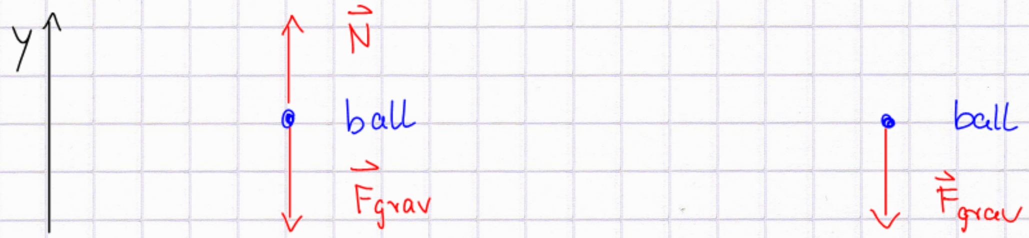
- $\frac{d}{dt}\left(f(t) + g(t)\right) = \frac{df}{dt} + \frac{dg}{dt}$
- $\frac{d}{dt}\left(\alpha f(t)\right) = \alpha f(t)$ (for any $\alpha \in \mathbb{R}$)
- $\frac{d}{dt} t^n = n\, t^{n-1}$

Analysis

- $F_{x,net} = 0$ at all times: Newton-1 applies for horizontal motion (if frictionless)

- Gravity (weight) $\vec{F}_{grav}$ acts in negative y direction
- $\vec{F}_{grav}$ is compensated by normal force $\vec{N}$ (due to table) at $t < 0$
- Free-body diagrams (FBDs)

$t < 0$ $\qquad\qquad\qquad\qquad$ $t > 0$



$$\vec{F}_{net} = \vec{F}_{grav} + \vec{N} = 0 \qquad\qquad \vec{F}_{net} = \vec{F}_{grav} = -mg\,\hat{j}$$

$$\Leftrightarrow \quad (-mg + N)\,\hat{j} = 0$$

$$\Leftrightarrow \quad N = mg$$

- Horizontal and vertical motions are independent and can be analyzed separately.

example 2: spacecraft in interstellar space

assume $\vec{F}_{grav} = 0$

case (i): engine turned off $\Rightarrow$ spacecraft is freely coasting on straight-line trajectory

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$

case (ii): engine turned on at $t = 0$ $\Rightarrow$ spacecraft is accelerating

assume
- $\vec{a} = a\,\hat{i}$ (for $t \geq 0$)
- $\vec{v}_0 = \vec{v}(t=0) = v_0\,\hat{i}$

infer : $\vec v(t) = (v_0 + at)\,\hat\imath$

$\boxed{\text{proof: } \frac{d\vec v}{dt} = a\hat\imath \ \checkmark}$

assume : $\vec r(t=0) = x_0\,\hat\imath$

(educated) guess : $\vec r(t) = \left(x_0 + v_0 t + \frac{a}{2}t^2\right)\hat\imath = x(t)\,\hat\imath$

check , $\frac{d\vec r}{dt} = (v_0 + at)\,\hat\imath = \vec v(t)$ ✓

For 1D motion with constant acceleration one can easily derive a direct relation between x and v from the results given above :

$v = v_0 + at \iff t = \frac{v - v_0}{a}$

insert

$$
\begin{aligned}
x &= x_0 + v_0 t + \frac{a}{2}t^2 \\
&= x_0 + v_0\frac{v - v_0}{a} + \frac{a}{2}\left(\frac{v - v_0}{a}\right)^2 \\
&= x_0 + \frac{v_0 v}{a} - \frac{v_0^2}{a} + \frac{a}{2}\frac{(v - v_0)^2}{a^2} \\
&= x_0 + \frac{1}{a}\left(\cancel{v_0 v} - v_0^2 + \frac{1}{2}\left(v^2 - \cancel{2v_0 v} + v_0^2\right)\right) \\
&= x_0 + \frac{v^2 - v_0^2}{2a}
\end{aligned}
$$

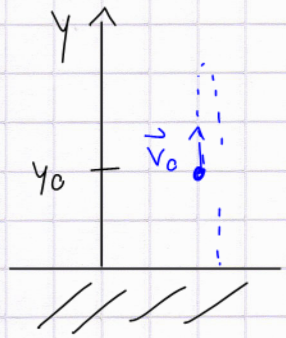
i.e. $x - x_0 = \frac{v^2 - v_0^2}{2a} \iff v^2 = v_0^2 + 2a(x - x_0)$

typical question that can be answered with this formula :

consider spacecraft with $v_0 = 100\ m/s$, $a = 20\ m/s^2$, $v_f = 400\ m/s$

- distance travelled ? $d = x_f - x_0 = \frac{v_f^2 - v_0^2}{2a} = \frac{400^2 - 100^2}{2\cdot 20}\frac{m^2 s^{-2}}{m s^{-2}} = 3.8\ km$
- how long did it take ? $v_f = v_0 + at_f \iff t_f = \frac{v_f - v_0}{a} = \frac{300\ ms^{-1}}{20\ ms^{-2}} = 15\,s$

example 3: free fall



FBD

$\vec{F}_{grav} = F_{grav}\,\hat{j}$

Newton-2: $a_y = \frac{F_{grav}}{m} = \frac{-mg}{m} = -g$

$$\left.\begin{aligned} \rightarrow\ v_y &= v_0 - gt \\ \rightarrow\ y &= y_0 + v_0 t - \frac{g}{2}t^2 \\ v_y^2 &= v_0^2 - 2g(y - y_0) \end{aligned}\right\}$$

see previous example with $a = -g$

typical questions and answers

- maximum height $\rightarrow$ characterized by $v_y = 0$

$$\rightarrow \quad v_y^2 = 0 = v_0^2 - 2g(y_{max} - y_0)$$

$$\Leftrightarrow \quad y_{max} = y_0 + \frac{v_0^2}{2g}$$

- how long does it take the ball to get there?

$$v_y(t_{max}) = v_0 - g\,t_{max} \overset{!}{=} 0$$

$$\Leftrightarrow \quad t_{max} = \frac{v_0}{g}$$

- when does the ball hit the ground?

$$y(t_{gr}) = y_0 + v_0 t_{gr} - \frac{g}{2}t_{gr}^2 \overset{!}{=} 0$$

$$\Leftrightarrow \quad t_{gr}^2 - \frac{2v_0}{g}t_{gr} - \frac{2y_0}{g} = 0$$

solve this quadratic equation for the special case $y_0 = 0$:

$$t_{gr}^2 - \frac{2v_0}{g}t_{gr} = t_{gr}\left(t_{gr} - \frac{2v_0}{g}\right) = 0$$

($t_{gr} = 0$ refers to initial situation)

$$\rightarrow \quad t_{gr} = \frac{2v_0}{g} = 2\,t_{max}$$

- velocity @ t_{gr}: $v_y(t_{gr}) = v_0 - g\,t_{gr} = v_0 - 2v_0 = -v_0$