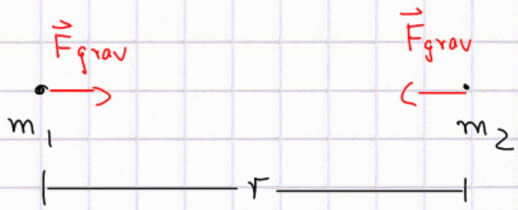
Gravitation and planetary motion

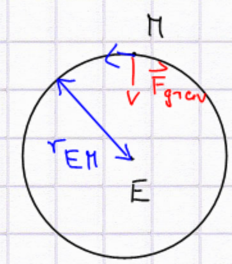
Newton also discovered the fundamental law of gravity



- $$F_{grav} = \frac{G m_1 m_2}{r^2} \quad \text{with} \quad G = 6.67 \times 10^{-11} \frac{Nm^2}{kg}$$
  "universal gravitational constant"
- $\vec{F}_{grav}$ directed along line that connects both bodies
- $\vec{F}_{grav}$ is always attractive
- m_1, m_2 are "point masses" (i.e. size $\ll r$)
- Newton-3 holds

example ①: Earth and Moon

Moon's orbit is almost perfectly circular.



Hence:
$$M_{Moon}\, a = M_{Moon}\, a_c = \frac{M_{Moon} v^2}{r_{EM}}$$
$$= F_{grav} = G \frac{M_{Moon} M_E}{r_{EM}^2}$$
$$\Leftrightarrow \quad v^2 = \frac{G M_E}{r_{EM}} \qquad (*)$$

Actually, the speed of the Moon can be obtained from
$$v = \omega r_{EM} = \frac{2\pi r_{EM}}{T} = 1.0 \times 10^3 \; m/s$$
(for known values of r_{EM} and T)

Use $(*)$ to determine $M_E = \frac{v^2 r_{EM}}{G} = 6.0 \times 10^{24}$ kg

If one also knows M_{Moon} (how can you measure it?), one can calculate $F_{grav} = G \frac{M_E M_{Moon}}{r_{EM}^2} = 2.0 \times 10^{20}$ N

That's a huge number, but gravity is in fact a relatively weak force:

example ②: two persons (this is just an estimate since they cannot really be considered "point particles")

$r = 20$ m
$m_1 = m_2 = 70$ kg $\;\longrightarrow\;$ $F_{grav} = 8 \times 10^{-10}$ N

(ridiculously small!)

Q: Why did we ignore the attraction on Earth due to Moon in example ①?

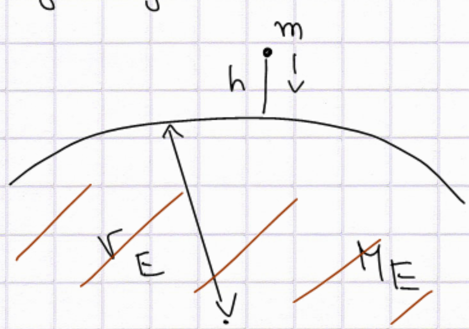
A: Earth is too heavy to be very impressed ($M_E \gg M_{Moon}$)

Q: Do we now have two laws of gravitation;

$$F^{(1)}_{grav} = \frac{G m_1 m_2}{r^2}, \quad F^{(2)}_{grav} = mg \quad ?$$

When Newton was struck by the famous apple he realized: No!

free fall reconsidered



Math (calculus) exercise:

F_{grav} (sphere w/ constant $\rho = \frac{M}{V}$) = F_{grav} (point mass with M @ centre)

$$\longrightarrow \quad F^{E\text{ on rock}}_{grav} = \frac{G M_E m}{(r_E + h)^2} \approx \frac{G M_E m}{r_E^2} \overset{!}{=} mg$$

The two laws are equivalent for near-Earth free-fall problem,

if: $$g = \frac{GM_E}{r_E^2} \qquad \text{(check it!)}$$

Then: for free-fall experiment on Moon one finds

$$F_{grav} = m\, g_{Moon} \quad \text{with} \quad g_{moon} = \frac{GM_{Moon}}{r_{Moon}^2} \approx \frac{g_{Earth}}{6}$$

Kepler's laws describe planetary motion. They can be inferred from Newton's law of gravity and his laws of motion, but Kepler found them through observations even before Newton was born. They are discussed in Chap 5.4 of the book. Below is just a very condensed summary.

I. Planets move on elliptical orbits

II. Equal areas law is fulfilled

III. $\text{Period}^2 \propto \text{major axis}^3$

We can prove III for special case of circular orbit (major axis $\to$ radius r)
(actually we've done it already on pp 1)

$$a = a_C = \frac{v^2}{r} = \frac{GM_\odot}{r^2} \iff v^2 = \frac{GM_\odot}{r} = \left(\frac{2\pi r}{T}\right)^2$$

$$\iff T^2 = \frac{4\pi^2}{GM_\odot} r^3$$

This equation can be used to determine $M_\odot$ (or in general: mass of the object in centre of circular orbit of another object)