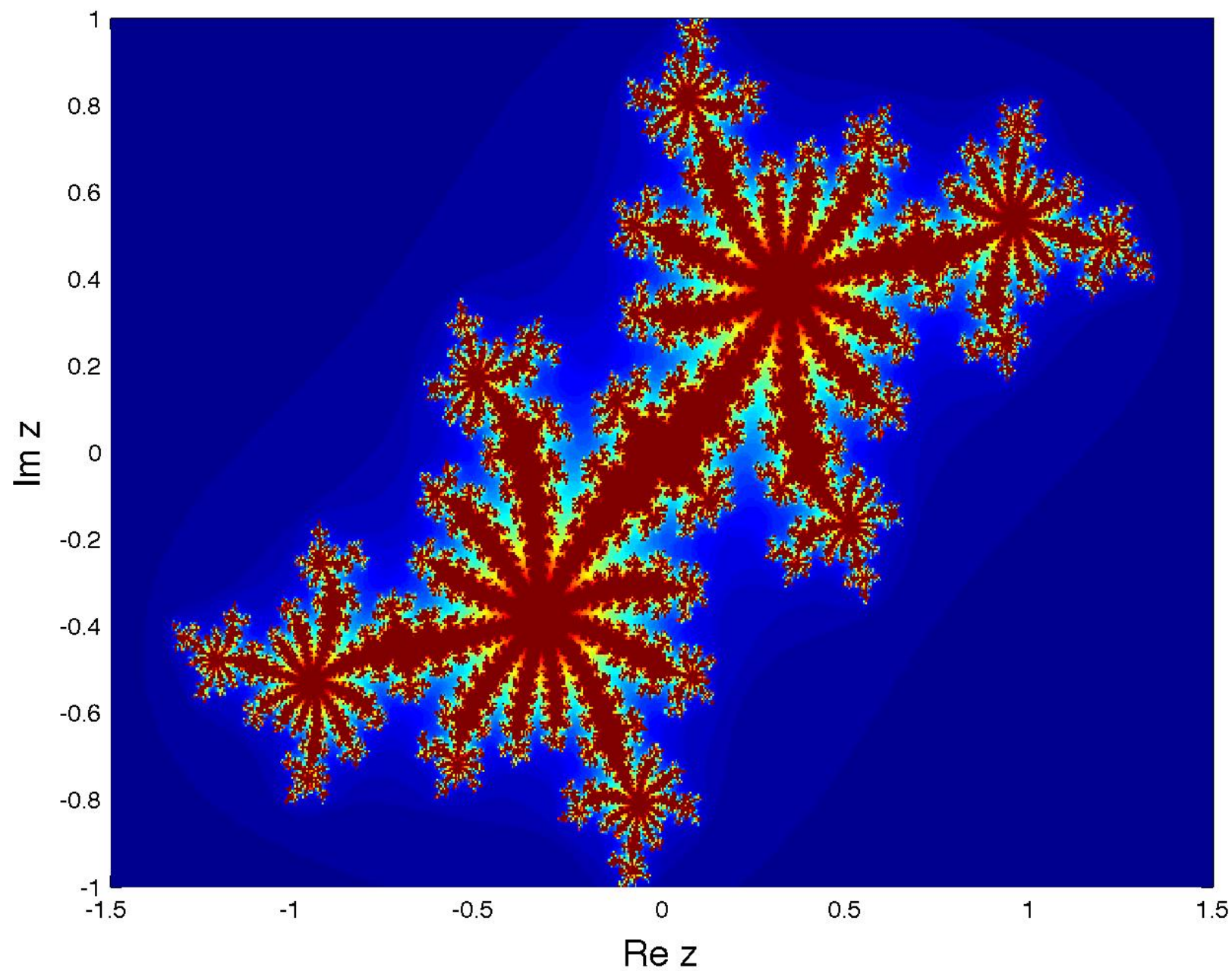


## **Computational Methods** (PHYS 2030)

**Instructors:** Prof. Christopher Bergevin (cberge@yorku.ca)

**Schedule:** Lecture: MWF 11:30-12:30 (CLH M)

**Website:** <http://www.yorku.ca/cberge/2030W2018.html>

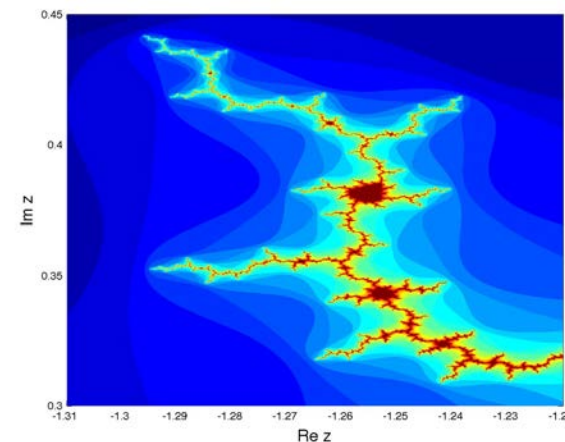
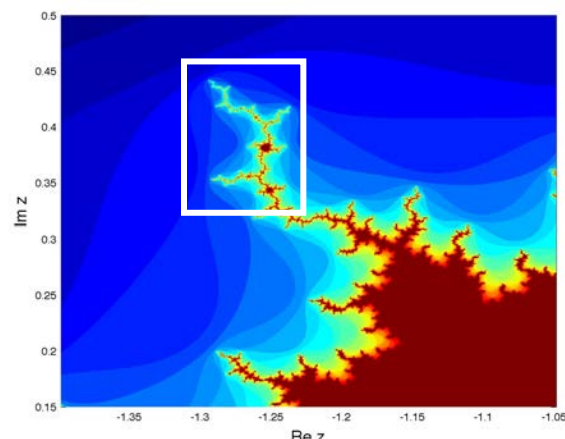
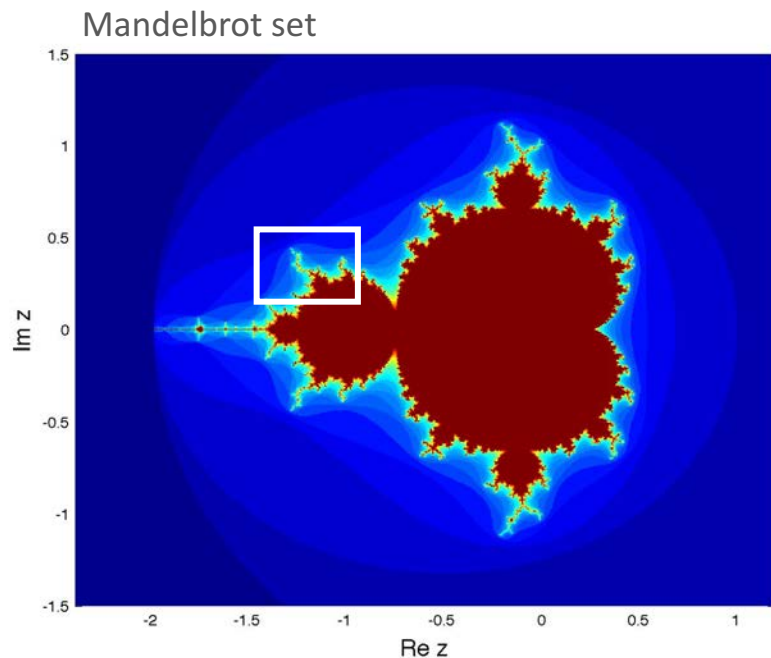






# What is a fractal?

- Consider a *monster curve*: Everywhere continuous, yet nowhere differentiable!
- Think of *self-similarity*: “A **fractal** is a natural phenomenon or a mathematical set that exhibits a repeating pattern that displays at every scale”

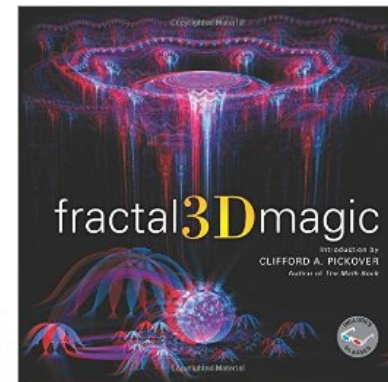




## mandelbrot set

**d**avid Darling writes that the Mandelbrot set, or M-set for short, is the “best known fractal and one of the most...beautiful mathematical objects known.” *The Guinness Book of World Records* called it “the most complicated object in mathematics.” Arthur C. Clarke emphasizes the degree to which the computer is useful for gaining insight: “In principle, [the Mandelbrot set] could have been discovered as soon as men learned to count. But even if they never grew tired, and never made a mistake, all the human beings who have ever existed would not have sufficed to do the elementary arithmetic required to produce a Mandelbrot set of quite modest magnification.”

The Mandelbrot set is a fractal, an object that continues to exhibit similar structural details no matter how much the edge of the object is magnified. Think of the beautiful M-set images as being produced by mathematical feedback loops. In fact, the set is produced by iteration, or repetition, of the very simple formula  $z_{n+1} = z_n^2 + c$ , for complex values of  $z$  and  $c$ , and for  $z_0 = 0$ . The set contains all points for which the formula does not produce values that diverge to infinity. The first crude pictures of the M-set were drawn in 1978 by Robert Brooks and Peter Matelski, followed by the landmark paper by Mandelbrot in 1980 on its fractal aspects and the wealth of geometric and algebraic information it conveys.



An (arbitrary) starting point.....

→ You have already seen some of the basic pieces needed (remember *sequences* from 1<sup>st</sup> year calculus?)

**Fixed-point iteration** (*iterare* is Latin for ‘to plow once again’)

$$x = F(x)$$

$F$  is assumed to be a differentiable function whose value can be computed for any given value of a real variable  $x$ , within a certain interval

$$x_1 = F(x_0), \quad x_2 = F(x_1), \quad x_3 = F(x_2), \dots \quad \text{Start with } x_0 \text{ \& iterate}$$

Or more generally:

$$x_{n+1} = F(x_n), \quad n = 0, 1, 2, \dots$$

If the sequence  $\{x_n\}$  converges to a limiting value  $\alpha$ , then we have

$$\alpha = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} F(x_n) = F(\alpha),$$

# introduction: what is a fractal?

**t**oday, computer-generated fractal patterns are everywhere. From squiggly designs on computer art posters to illustrations in the most serious of physics journals, interest continues to grow among scientists and, rather surprisingly, artists and designers. The word fractal was coined in 1975 by mathematician Benoît Mandelbrot to describe an intricate-looking set of curves, many of which were never seen before the advent of computers with their ability to quickly perform massive calculations. Fractals often exhibit self-similarity, which suggests that various exact or inexact copies of an object can be found in the original object at smaller size scales. The detail continues for many magnifications—

like an endless nesting of Russian dolls within dolls. Some of these shapes exist only in abstract geometric space, but others can be used as models for complex natural objects such as coastlines and blood vessel branching. These dazzling computer-generated images can be intoxicating, motivating students' interest in math more than any other mathematical discovery in the last century.

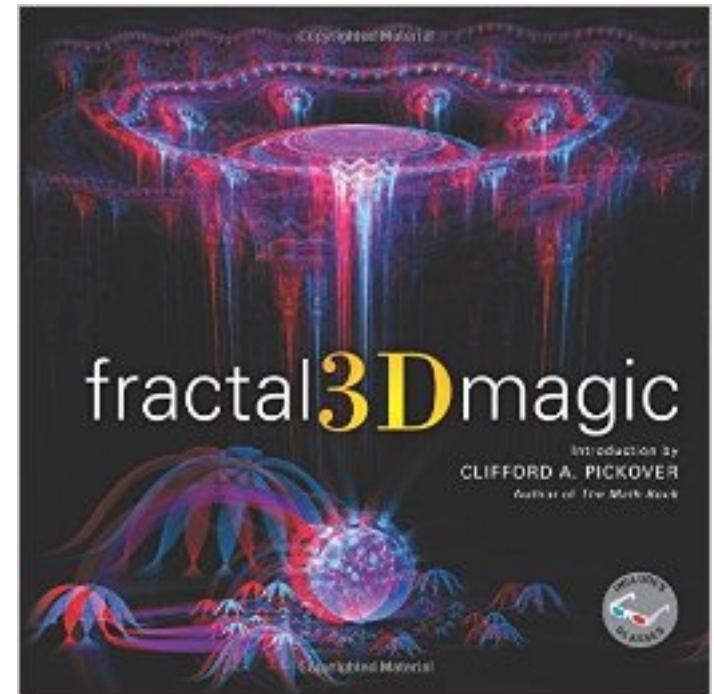
Physicists are interested in fractals because they can sometimes describe the chaotic behavior of real-world phenomena such as planetary motion, fluid flow, the diffusion of drugs, the behavior of inter-industry relationships, and the vibration of airplane wings. (Chaotic behavior often produces

fractal patterns.) Traditionally, when physicists or mathematicians saw complicated results, they looked for complicated causes. In contrast, many fractal shapes reveal the fantastically complicated behavior of the simplest formulas.

Early explorers of fractal objects include Karl Weierstrass, who in 1872 considered functions that were everywhere continuous but nowhere differentiable, and Helge von

Koch, who in 1904 discussed geometric shapes such as the Koch Snowflake. In the nineteenth and early twentieth centuries, several mathematicians explored fractals in the complex plane; however, they could not fully appreciate or visualize these objects without the aid of the computer.<sup>1</sup>

Clifford A. Pickover  
Yorktown Heights, NY





## Fractals in nature

- Are fractals just some sort of mathematical oddity?

**No** → Many real-world aspects exhibit fractal geometry

Romanesco broccoli

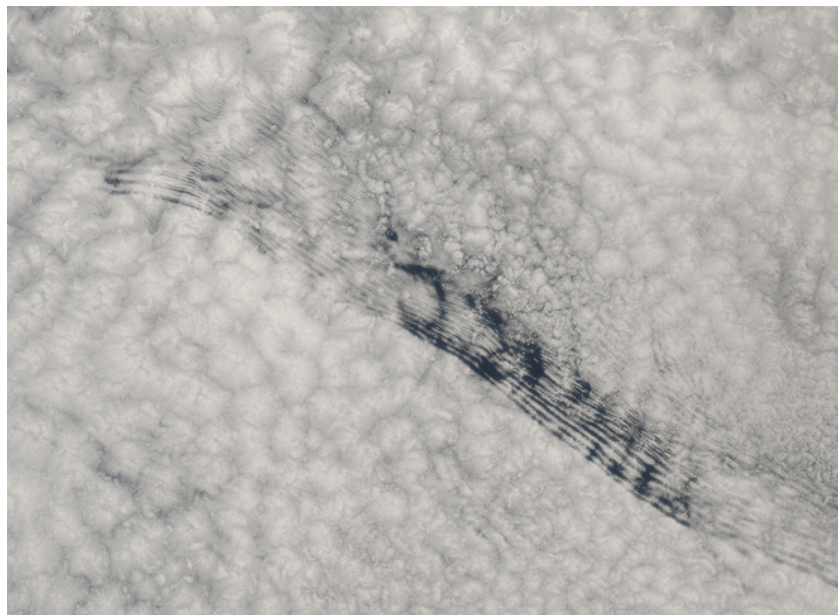


Frost





# Fractals in nature



Clouds

Leaves & plants



Mountains, coastlines, & river deltas



Lightning



# Reports

## How Long Is the Coast of Britain?

### Statistical Self-Similarity and Fractional Dimension

**Abstract.** *Geographical curves are so involved in their detail that their lengths are often infinite or, rather, undefinable. However, many are statistically "self-similar," meaning that each portion can be considered a reduced-scale image of the whole. In that case, the degree of complication can be described by a quantity  $D$  that has many properties of a "dimension," though it is fractional; that is, it exceeds the value unity associated with the ordinary, rectifiable, curves.*

Seacoast shapes are examples of highly involved curves such that each of their portion can—in a statistical sense—be considered a reduced-scale image of the whole. This property will be referred to as "statistical self-similarity." To speak of a length for such figures is usually meaningless. Similarly (1), "the left bank of the Vistula, when measured with increased precision, would furnish lengths ten, hundred or even thousand times as great as the length read off the school map." More generally, geographical curves can be

considered as superpositions of features of widely scattered characteristic size; as ever finer features are taken account of, the measured total length increases, and there is usually no clearcut gap between the realm of geography and details with which geography need not be concerned.

Quantities other than length are thus needed to discriminate between various degrees of complication for a geographical curve. When a curve is self-similar, it is characterized by an exponent of similarity,  $D$ , which possesses

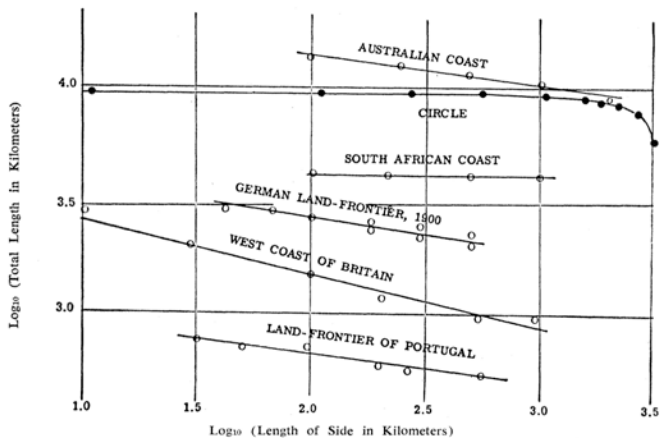


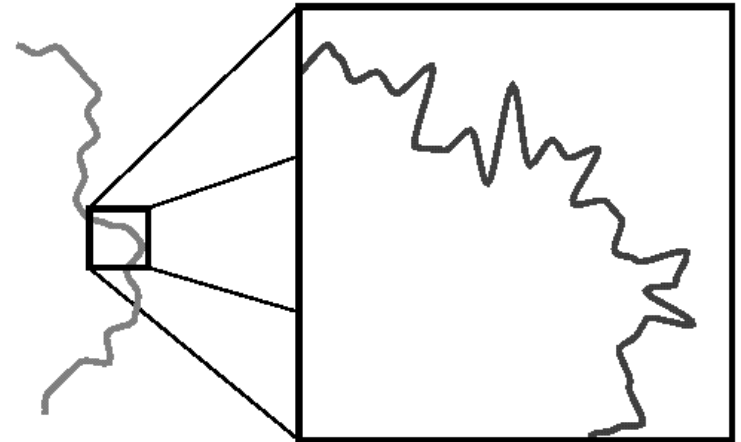
Fig. 1. Richardson's data concerning measurements of geographical curves by way of polygons which have equal sides and have their corners on the curve. For the circle, the total length tends to a limit as the side goes to zero. In all other cases, it increases as the side becomes shorter, the slope of the doubly logarithmic graph being in absolute value equal to  $D-1$ . (Reproduced from 2, Fig. 17, by permission.)

many properties of a dimension, though it is usually a fraction greater than the dimension 1 commonly attributed to curves. We shall reexamine in this light some empirical observations by Richardson (2). I propose to interpret them as implying, for example, that the dimension of the west coast of Great Britain is  $D = 1.25$ . Thus, the so far esoteric concept of "random figure of fractional dimension" is shown to have simple and concrete applications and great usefulness.

Self-similarity methods are a potent tool in the study of chance phenomena, including geostatistics, as well as economics (3) and physics (4). In fact, many noises have dimensions  $D$  contained between 0 and 1, so that the scientist ought to consider dimension as a continuous quantity ranging from 0 to infinity.

Returning to the claim made in the first paragraph, let us review the methods used when attempting to measure the length of a seacoast. Since a geographer is unconcerned with minute details, he may choose a positive scale  $G$  as a lower limit to the length of geographically meaningful features. Then, to evaluate the length of a coast between two of its points  $A$  and  $B$ , he may draw the shortest inland curve joining  $A$  and  $B$  while staying within a distance  $G$  of the sea. Alternatively, he may draw the shortest line made of straight segments of length at most  $G$ , whose vertices are points of the coast which include  $A$  and  $B$ . There are many other possible definitions. In practice, of course, one must be content with approximations to shortest paths. We shall suppose that measurements are made by walking a pair of dividers along a map so as to count the number of equal sides of length  $G$  of an open polygon whose corners lie on the curve. If  $G$  is small enough, it does not matter whether one starts from  $A$  or  $B$ . Thus, one obtains an estimate of the length to be called  $L(G)$ .

Unfortunately, geographers will disagree about the value of  $G$ , while  $L(G)$  depends greatly upon  $G$ . Consequently, it is necessary to know  $L(G)$  for several values of  $G$ . Better still, it would be nice to have an analytic formula linking  $L(G)$  with  $G$ . Such a formula, of an entirely empirical character, was proposed by Lewis F. Richardson (2) but unfortunately it attracted no attention. The formula is  $L(G) = M G^{1-D}$ , where  $M$  is a positive constant and  $D$  is a constant at least equal to unity. This











ORIGINAL ARTICLE

# Fractal Analysis of Eye Movements during Reading

**ELMAR T. SCHMEISSER, PhD, FAAO, J. M. MCDONOUGH, PhD, MARY BOND, MA,  
PETER D. HISLOP, PhD, and AVROM D. EPSTEIN, MD**

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**ABSTRACT:** We present a new method for the analysis of reading eye movements based on the methods of nonlinear dynamics. In this preliminary study, the eye movements of normal and abnormal readers were analyzed for evidence of chaotic, nonlinear dynamical behavior. Both power spectral density analysis and fractal dimension determination showed evidence of nonlinearity as manifest in chaotic behavior. The computed fractal dimension of the system's presumed attractor seemed directly related to qualitative assessment of reading ability. Representative subjects did not differ in a similar analysis of pursuit movements. Although fractal analysis did not distinguish unconditionally between normal and abnormal reading in this preliminary study, it nevertheless offers a promising hitherto unused approach to the analysis of eye movements and the modeling of oculomotor behavior. (Optom Vis Sci 2001;78:805–814)

**Key Words:** eye movement, reading, pursuit, nonlinear dynamical systems, fractals

## ORIGINAL ARTICLE

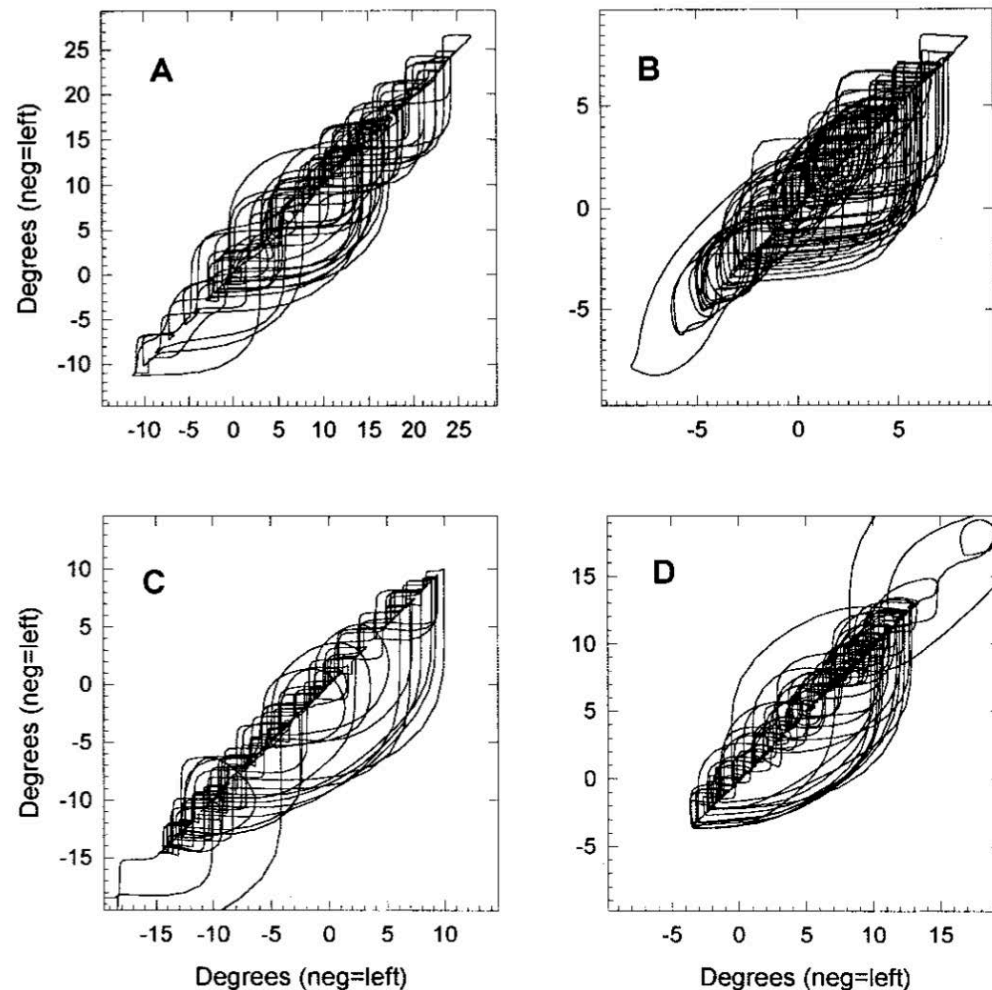
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**Key Words:** eye movement, reading, pursuit, nonlinear dynamical systems, fractals



**FIGURE 3.**

Phase space plots (delay maps) of the Shuman filtered reading traces shown in Fig. 1. The panels are the same as in Fig. 1.



## Fractal character of fracture surfaces of metals

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When a piece of metal is fractured either by tensile or impact loading (pulling or hitting), the fracture surface that is formed is rough and irregular. Its shape is affected by the metal's microstructure (such as grains, inclusions and precipitates, whose characteristic length is large relative to the atomic scale), as well as by 'macrostructural' influences (such as the size, the shape of the specimen, and the notch from which the fracture begins). However, repeated observation at various magnifications also reveals a variety of additional structures that fall between the 'micro' and the 'macro' and have not yet been described satisfactorily in a systematic manner. The experiments reported here reveal the existence of broad and clearly distinct zone of intermediate scales in which the structure is modelled very well by a fractal surface. A new method, slit island analysis, is introduced to estimate the basic quantity called the fractal dimension,  $D$ . The estimate is shown to agree with the value obtained by fracture profile analysis, a spectral method. Finally,  $D$  is shown to be a measure of toughness in metals.

Fractals are the concern of a new geometry<sup>1</sup>, whose primary object is to describe the great variety of natural structures that are irregular, rough or fragmented, having irregularities of various sizes that bear a special 'scaling' relationship to one another. In very loose terms, they seem to fall into a regular hierarchy in which each level is an up-sized or down-sized version of the levels below or above it. Fractal geometry characterizes the scaling structure of a surface by a number  $D$ , called the fractal dimension, that can range from 2, when the surface is smooth, up to 3. The fractal dimensional increment of a surface is  $D-2$ . As it increases from 0 to 1, the irregularities become increasingly predominant and the notion of overall shape of the surface becomes progressively less meaningful. Similarly, the fractal dimension of a curve is at least 1, the fractal dimensional increment being  $D-1$ .

The term 'fractal' was chosen in explicit cognizance of the fact that the irregularities found in fractal sets are often strikingly reminiscent of fracture surfaces in metals (though not, for example in glass). However, metal fractures are only extremely crinkly (down to the limits of their microstructural size range), while fractals are infinitely crinkly. Hence, it is not possible to say that metal fracture is strictly a fractal. Nevertheless, metals resemble a fractal so closely that it makes good sense to use a model for modelling a fractal. Our experiments in metal fracture show that  $D$  is very well defined for all the specimens examined, and takes on the same value for different specimens of the same metal having similar thermomechanical treatments.

Recent studies<sup>2,3</sup> conclude that a metal fracture's spectrum is indicative of a fractal structure. A recent review<sup>4</sup> also indicates that fractals might model metal fracture surfaces, and reports exploratory studies. The more refined tools that we use to analyse metal fracture surfaces make a stronger case and add to the existing tools of fractographic analysis<sup>5,6</sup>.

Our first series of tests is exemplified by Fig. 1. A fractured steel specimen was plated with electroless nickel and mounted in an epoxy mount by vacuum impregnation in order to ensure edge retention. The specimen was then polished parallel to the plane of fracture. 'Islands' of steel surrounded by nickel appeared which, on subsequent polishing, grew and merged. We propose to call these structures slit islands. The islands

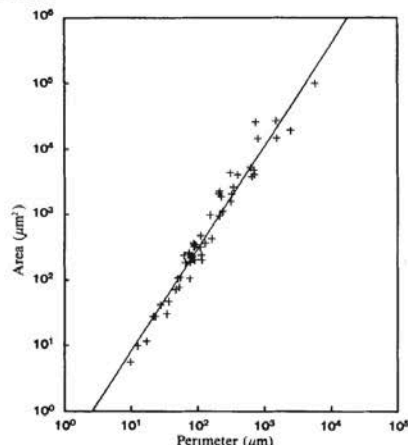


Fig. 1 Fractal area-perimeter relationship for slit islands. 300-Grade Maraging steel. Ruler = 1.5625  $\mu\text{m}$ ; fractal dimensional increment = 1.28.

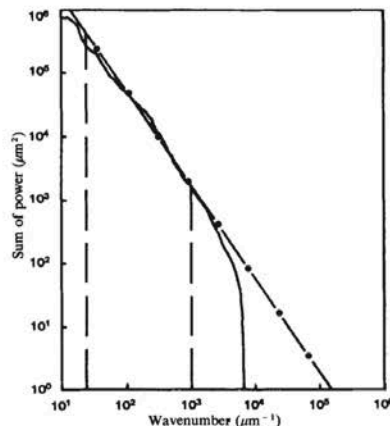


Fig. 2 Cumulative spectrum for vertical cuts. 300-Grade Maraging steel. Slope = 1.4878; fractal dimensional increment = 1.26. Average of five profiles.

contain 'lakes within islands' and 'islands within lakes'; we include the former and neglect the latter.

The slit island 'coastlines', being curves, are much easier to investigate than the surfaces themselves. Yet we have found that the perimeter-area relationship of fractal dimensional analysis (ref. 1, Chap. 12), if applied to these coastlines, reveals many essential facts about the surface. We propose that the resulting method of analysis, which is entirely new and very powerful, be called slit island analysis. When islands are derived from an initial fractal surface of dimension  $D$  by sectioning with a plane, their coastlines are of fractal dimension  $D' = D-1$ . Thus, the fractal dimensional increments  $D-2$  and  $D'-1$  are equal. The theory of fractals suggests that the areas and

"When a piece of metal is fractured either by tensile or impact loading (pulling or hitting), the fracture surface that is formed is rough and irregular. Its shape is affected by the metal's microstructure (such as grains, inclusions and precipitates, whose characteristic length is large relative to the atomic scale), as well as by 'macrostructural' influences (such as the size, the shape of the specimen, and the notch from which the fracture begins). However, repeated observation at various magnifications also reveals a variety of additional structures that fall between the 'micro' and the 'macro' and have not yet been described satisfactorily in a systematic manner. The experiments reported here reveal the existence of broad and clearly distinct zone of intermediate scales in which the structure is modelled very well by a fractal surface. A new method, slit island analysis, is introduced to estimate the basic quantity called the fractal dimension,  $D$ . The estimate is shown to agree with the value obtained by fracture profile analysis, a spectral method. Finally,  $D$  is shown to be a measure of toughness in metals."

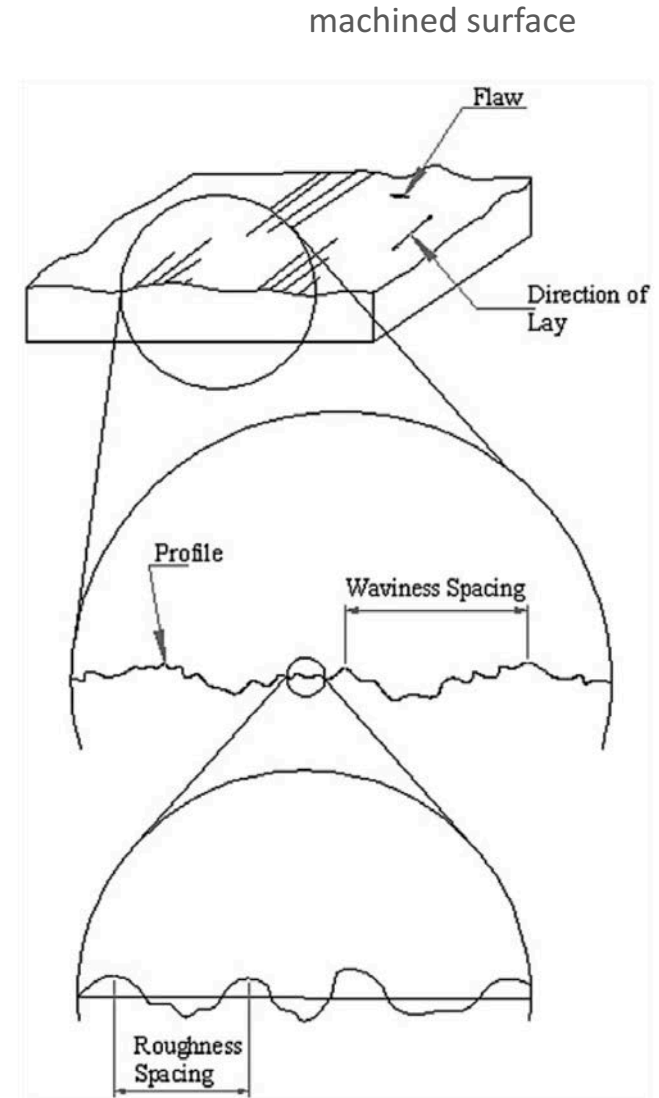
## Fractal character of fracture surfaces of metals

**Benoit B. Mandelbrot\***, **Dann E. Passoja†**  
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“Surfaces are irregular though they may look like very smooth. When the surfaces are magnified, the irregularities become prominent. This is true for the machining surfaces as well. In a material removal process such as machining, unwanted material is removed and altered surface topography is obtained. The surface generated consists of inherent irregularities left by the cutting tool, which are commonly defined as surface roughness. [...]

Surface roughness plays an important role. It has large impact on the mechanical properties like fatigue behavior, corrosion resistance, creep life, etc. It also affects other functional attributes of machine components like friction, wear, light reflection, heat transmission, lubrication, electrical conductivity, etc.”

**Fig. 1.1** Display of surface texture



# Fractal structure enables temporal prediction in music

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**Abstract:**  $1/f$  serial correlations and statistical self-similarity (fractal structure) have been measured in various dimensions of musical compositions. Musical performances also display  $1/f$  properties in expressive tempo fluctuations, and listeners predict tempo changes when synchronizing. Here the authors show that the  $1/f$  structure is sufficient for listeners to predict the onset times of upcoming musical events. These results reveal what information listeners use to anticipate events in complex, non-isochronous acoustic rhythms, and this will entail innovative models of temporal synchronization. This finding could improve therapies for Parkinson's and related disorders and inform deeper understanding of how endogenous neural rhythms anticipate events in complex, temporally structured communication signals.

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### (A) Pitch



### (B) Rhythm



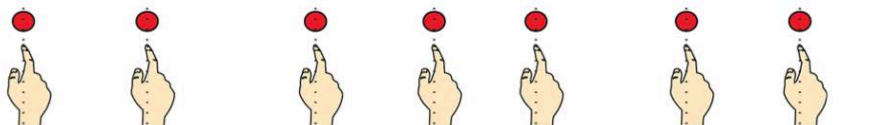
### (C) Eighth



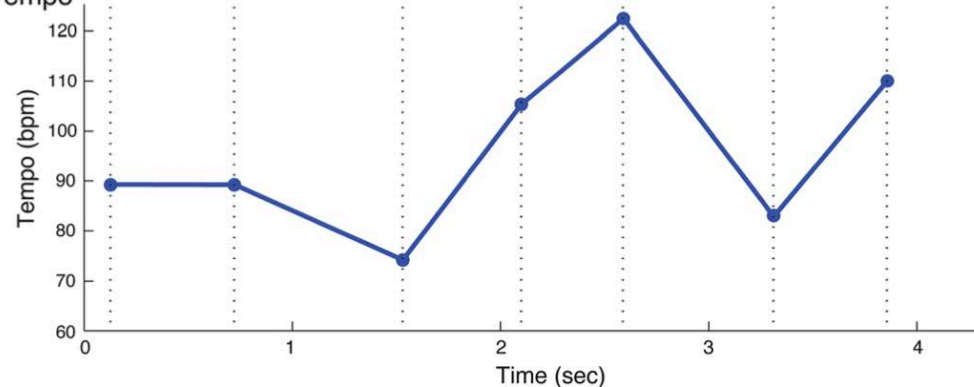
### (D) Quarter



### (E) Task



### (F) Tempo



## Hierarchical, self-similar structure in native squid pen

Cite this: *Soft Matter*, 2014, 10, 5541

Fei-Chi Yang,<sup>a</sup> Robert D. Peters,<sup>a</sup> Hannah Dies<sup>a</sup> and Maikel C. Rheinstädter<sup>\*ab</sup>

The structure of native squid pen (gladius) was investigated in two different species on different length scales. By combining microscopy, atomic force microscopy (AFM), and X-ray diffraction, the experiments probed length scales from millimetres down to nanometres. The gladii showed a hierarchical, self-similar structure in the optical experiments with fibres of different size oriented along the long axis of the gladius. The fibre-like structure was reproduced at the nanoscale in AFM measurements and fibres with diameters of 500  $\mu\text{m}$ , 100  $\mu\text{m}$ , 10  $\mu\text{m}$ , 2  $\mu\text{m}$  and 0.2  $\mu\text{m}$  were observed. Their molecular structure was determined using X-ray diffraction. In the squid gladius, the chitin molecules are known to form nanocrystallites of monoclinic lattice symmetry wrapped in a protein layer, resulting in  $\beta$ -chitin nano-fibrils. Signals corresponding to the  $\alpha$ -coil protein phase and  $\beta$ -chitin crystallites were observed in the X-ray experiments and their orientation with respect to the fibre-axis was determined. The size of a nano-fibril was estimated from the X-ray experiments to be about  $150 \times 300 \text{ \AA}$ . About 100 of these nano-fibrils are needed to form a 0.2  $\mu\text{m}$  thick micro-fibre. We found that the molecular structure is highly anisotropic with  $\sim 90\%$  of the  $\alpha$ -coils and  $\beta$ -chitin crystallites oriented along the fibre-axis, indicating a strong correlation between the macroscale structure and molecular orientation.

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[www.rsc.org/softmatter](http://www.rsc.org/softmatter)

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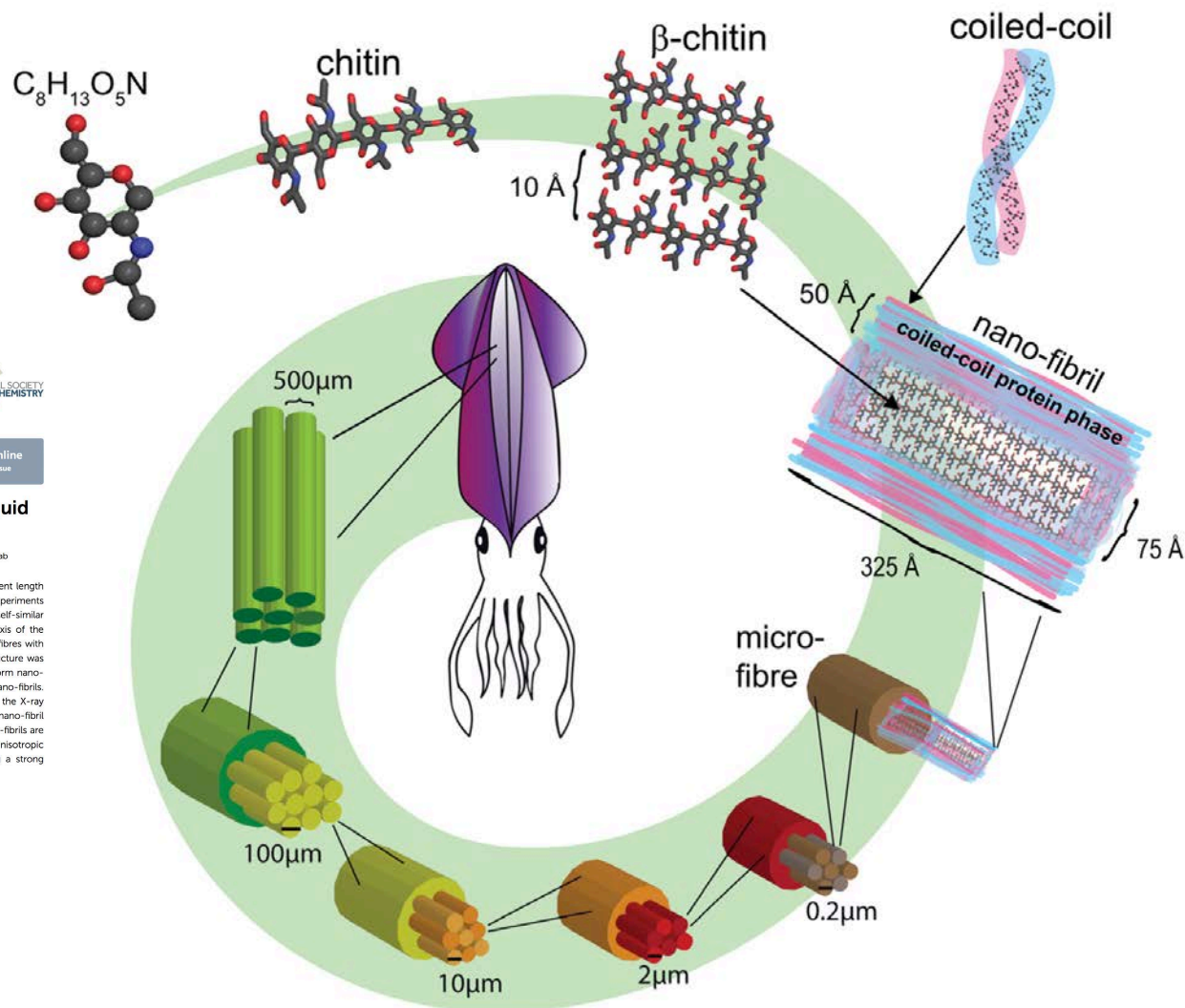
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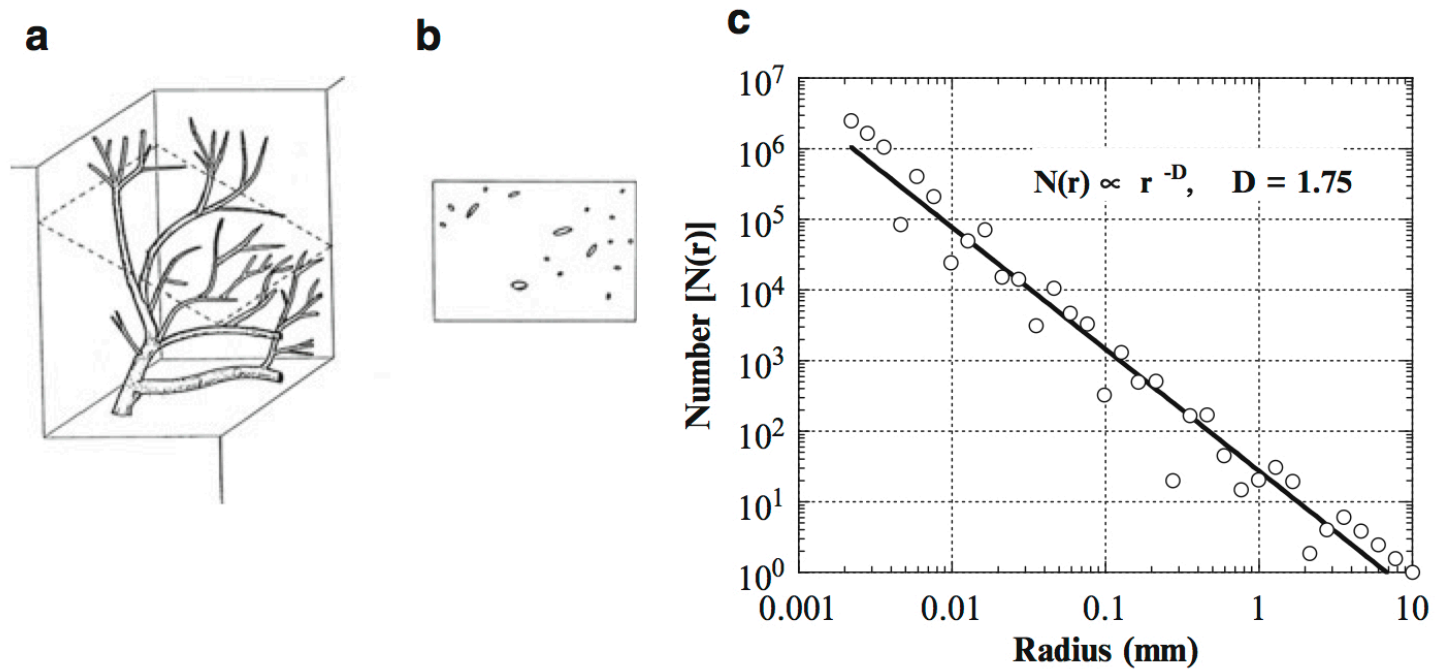
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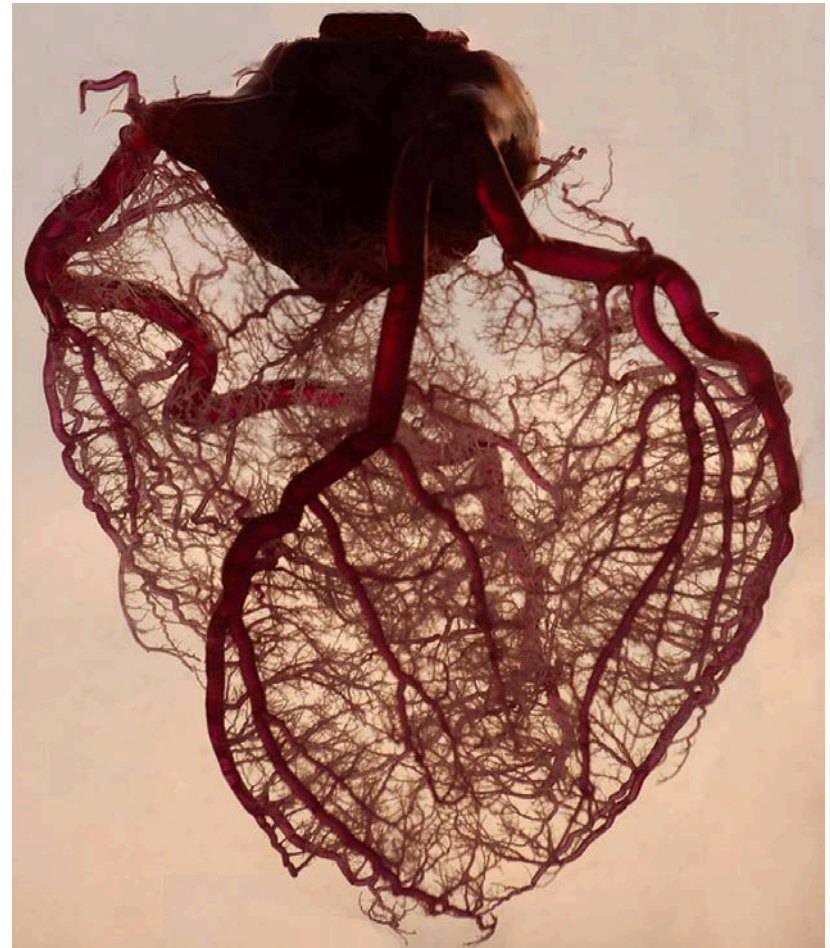






**Fig. 1.1** Schematic representations of the method for measuring a fractal dimension,  $D$ , of a natural branching system. (a) In situ vascular tree in the tissue specimen. (b) Cross section of the tissue including the circular and elliptic cut ends of vascular branches with various sizes. The minimum width of individual branch cross sections is used to estimate the radius of branch segments. (c) The number in the ordinate indicates the cumulative number of branch segments with radii larger than or equal to the variable  $r$ . The number of branches  $[\tilde{N}(r)]$  is plotted on a log–log scale. The negative of the slope of the power regression is the fractal dimension  $D$  for the fractal tree. The data points were obtained in part from the measurements in the rat brain vasculature by Matsuo et al. (1990). Adapted from Kamiya and Takahashi (2007), reproduced by permission

## Fractals in nature

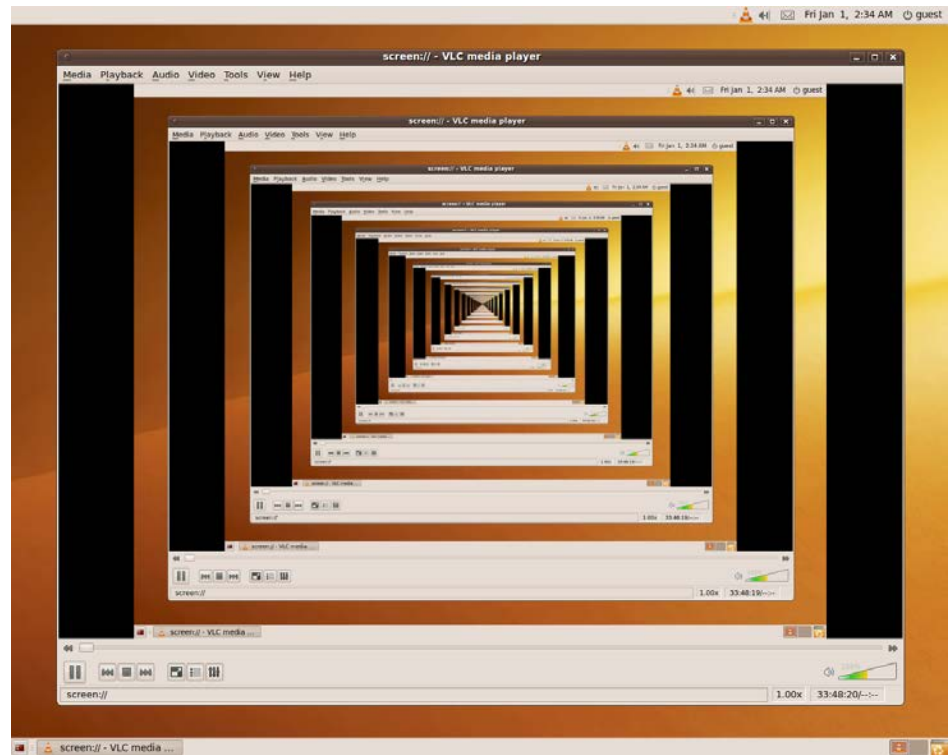


See also:

- <http://www.bodyworlds.com/en.html>
- <http://classes.yale.edu/fractals/>
- [http://www.ted.com/talks/benoit\\_mandelbrot\\_fractals\\_the\\_art\\_of\\_roughness](http://www.ted.com/talks/benoit_mandelbrot_fractals_the_art_of_roughness)

## How can we 'make' a fractal?

- A basic ingredient is some sort of *recursion* relation (i.e., some sort of computational 'production rule')
- Classic example comes from Helge von Koch (1904)
  - Koch's 'snowflake'



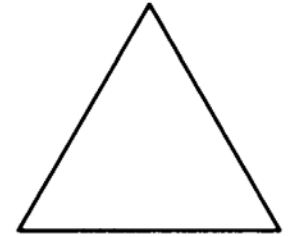
Let's imagine that you're instructing an incredibly stupid machine to draw the figure, so we want to keep the instructions as simple as possible. To describe the von Koch curve, we need only four instructions: *F*, to go forward one step; *+*, to turn to the right by 60°; *-*, to turn to the left by 60°; and *T*, to reduce the size of the step by a factor of one-third. To construct the von Koch curve, we begin with an equilateral triangle with sides of unit length. The instructions to draw this triangle are simply

$$F + + F + + F + + .$$



How can we 'make' a fractal?

$F ++ F ++ F ++$

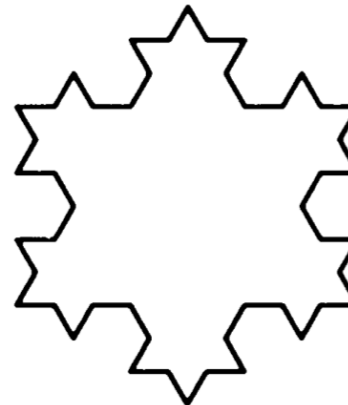
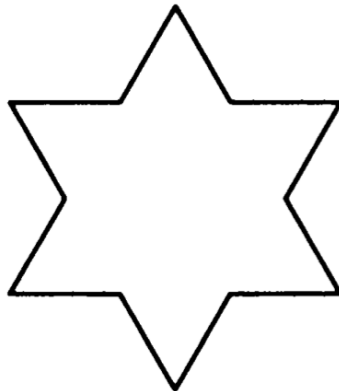


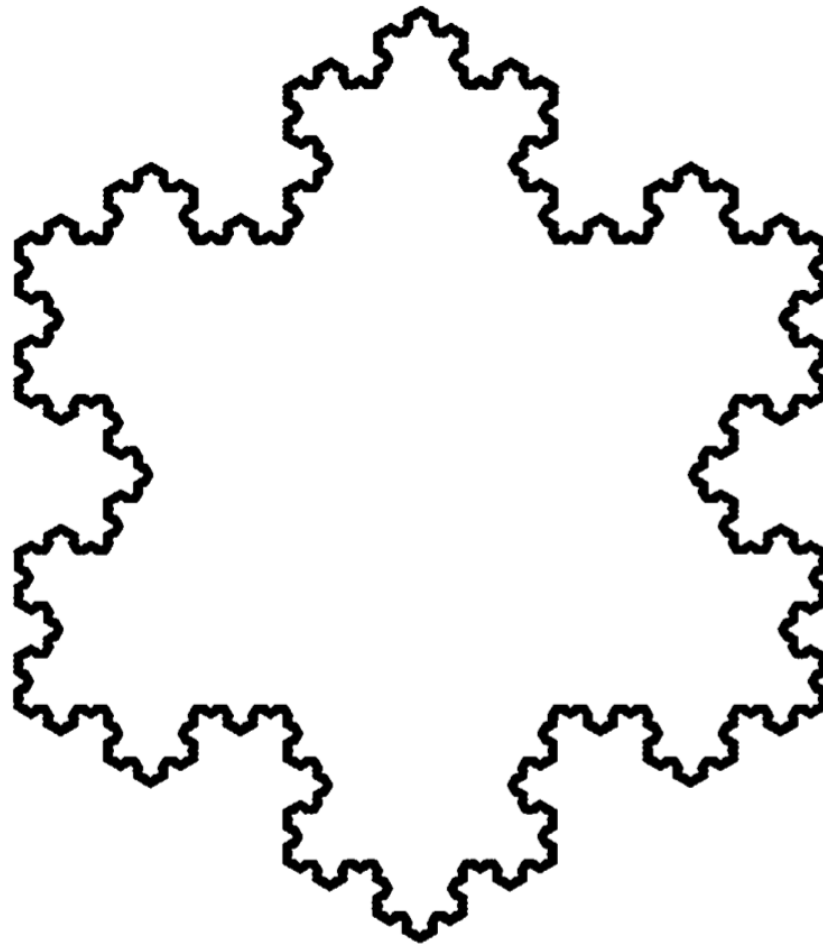
To produce a new figure, we follow two rules: first, add a  $T$  to the beginning of the instruction list; and second, replace  $F$  by a new set of instructions:

$$F \rightarrow F - F ++ F - F.$$

That is, every time an  $F$  appears in the original set of instructions, it is replaced by  $F - F ++ F - F$ . If we follow these rules, beginning with the original description of the figure, we produce a new list of instructions:

$$TF - F ++ F - F ++ F - F ++ F - F ++ F - F ++ F - F ++.$$





**FIGURE 1.4** The von Koch curve, a.k.a. the Snowflake.

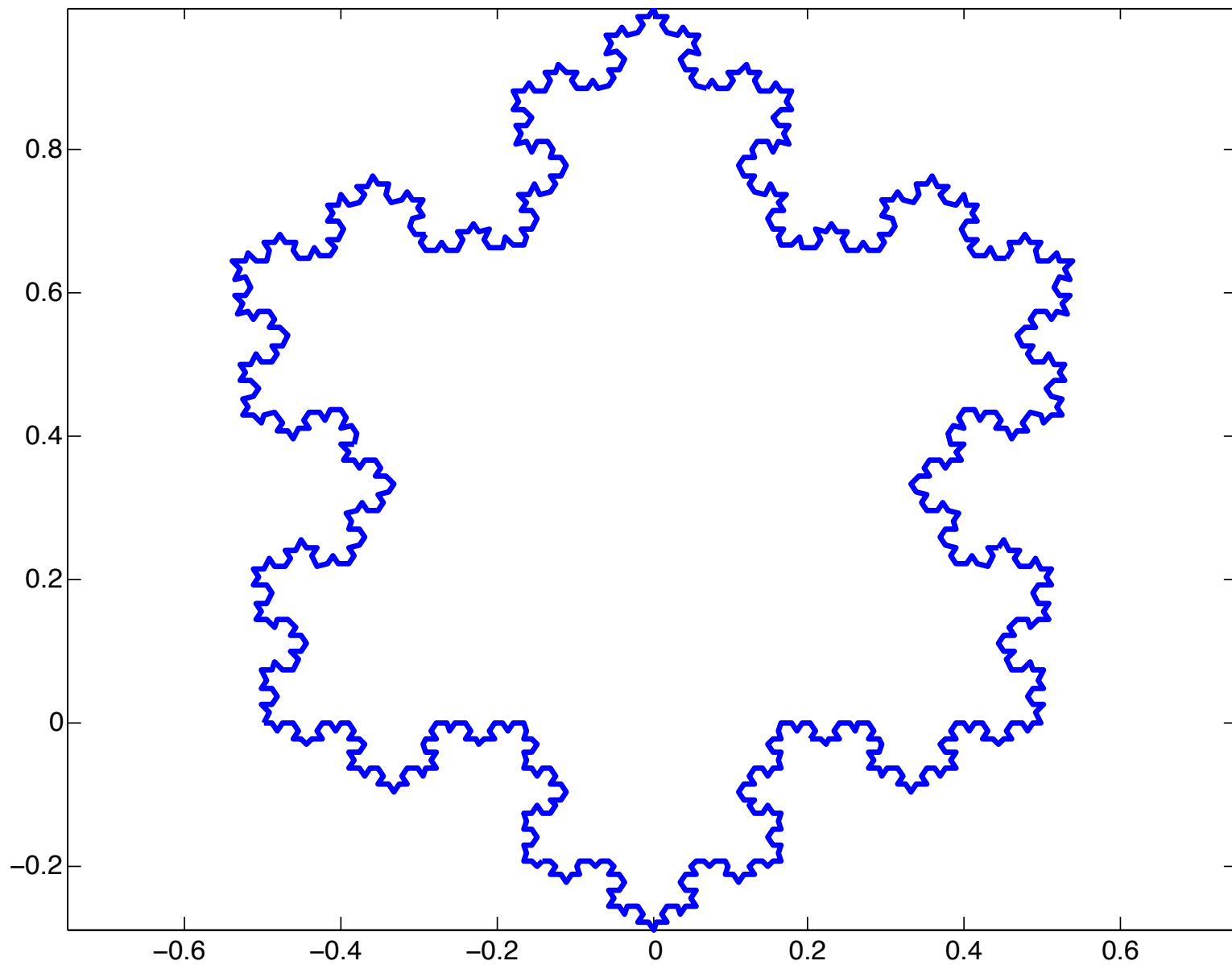
```

% ### EXkoch1.m ###      10.14.14
% Plot Koch's 'Snowflake' (see http://mathworld.wolfram.com/KochSnowflake.html)
% [source - http://www.math.columbia.edu/~fusco/Hmwk01\_APAM4300.pdf]
clear; clf;
% -----
% User Inputs
n = 2;      % 'order' of the snowflake
% -----
x = [-1/2 0 1/2 -1/2]; y = [0 1 0 0]; % starting matrices (for n=0)
% ---
% loop to create 'snowflake' - each time this outer loop executes, the
% inner loop updates to the next iteration
for zz = 1:n
    k = length(x);
    v = zeros(4*k-3,1); % dummies (v-->x, w-->y) to fill up in inner loop;
    w = zeros(4*k-3,1); % size stems from increase in resolution
    % this loop determines the 'iterated' version, each time updating based
    % upon the required rotations/reductions in step-size
    for j = 1:k-1,
        v(4*j-3) = x(j)
        w(4*j-3) = y(j);
        dirx = x(j+1) - x(j);
        diry = y(j+1) - y(j);
        v(4*j-2) = x(j) + 1/3*dirx;
        w(4*j-2) = y(j) + 1/3*diry;
        orthox = -diry;
        orthoy = dirx;
        v(4*j-1) = x(j) + 1/2*dirx + 1/3*1/2*sqrt(3)*orthox;
        w(4*j-1) = y(j) + 1/2*diry + 1/3*1/2*sqrt(3)*orthoy;
        v(4*j) = x(j) + 2/3*dirx;
        w(4*j) = y(j) + 2/3*diry;
    end
    v(4*k-3) = x(k); % bring back to starting point
    w(4*k-3) = y(k);
    x = v; y = w; % update x and v (re inner loop)
end
% ---
% visualize
plot(x,y)
axis([-0.75 0.75 -sqrt(3)/6 1])

```



n= 4;



→ Zooming in upon the snowflake....



## How can we 'make' a fractal?

- A basic ingredient is some sort of *recursion* relation
- What else goes into a fractal? → *Nonlinearity*
- Consider the (deceptively simple) *logistic equation* as we have already examined:

$$\frac{dP}{dt} = kP \left( 1 - \frac{P}{L} \right)$$

(e.g., population growth, 'carrying capacity')

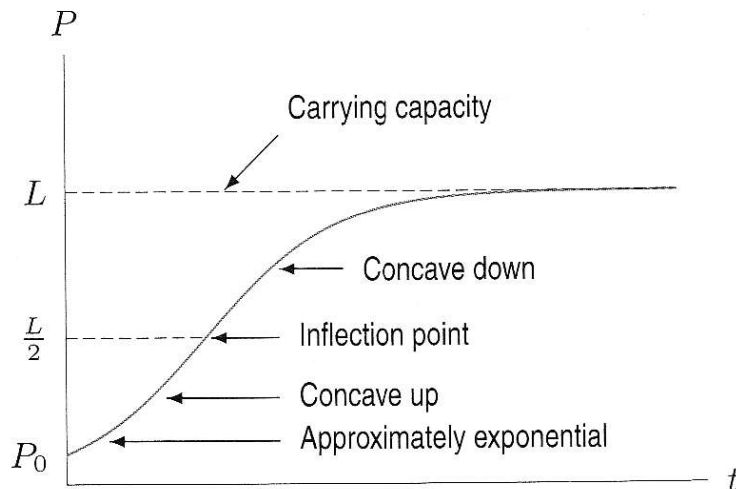


Figure 11.51: Logistic growth with inflection point

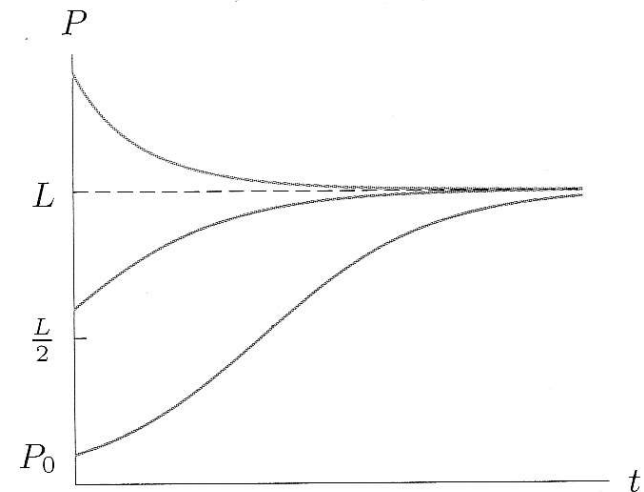
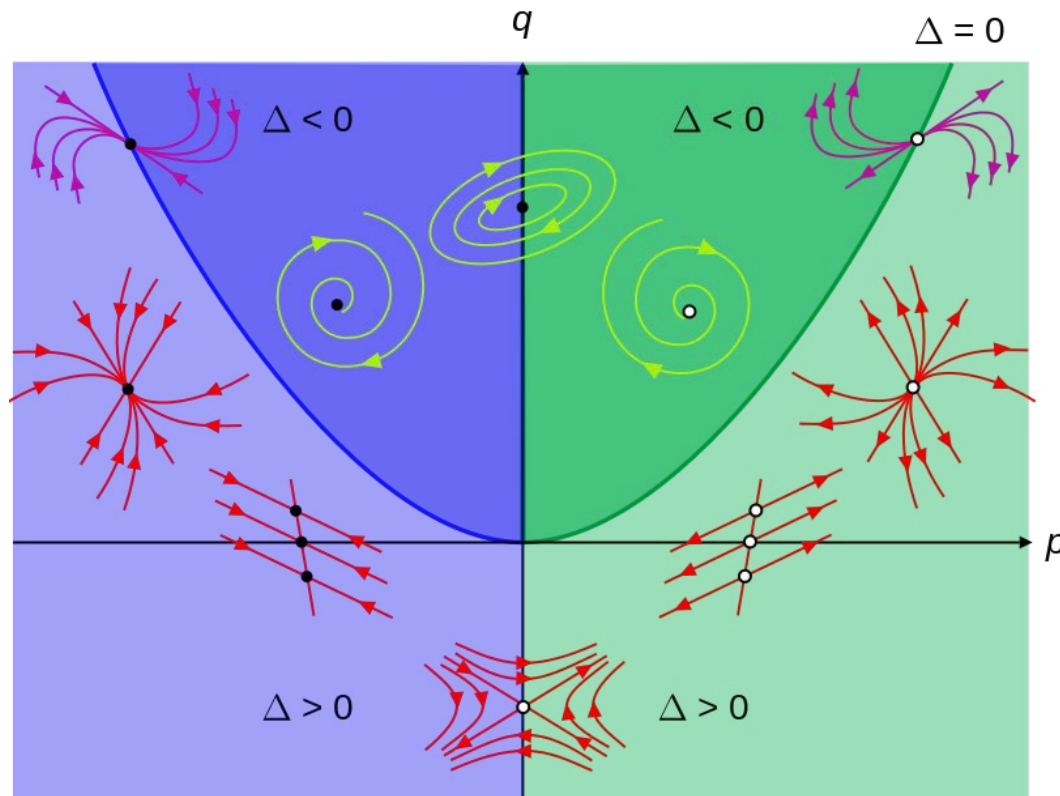


Figure 11.52: Solutions to the logistic equation



## Nonlinearity

- Seemingly simple nonlinearity has (surprisingly) vast implications!
- For a moment, consider that solutions to an autonomous system of linear 2<sup>nd</sup> order ODEs has a variety of interesting behaviors (as characterized by the associated eigenvalues):

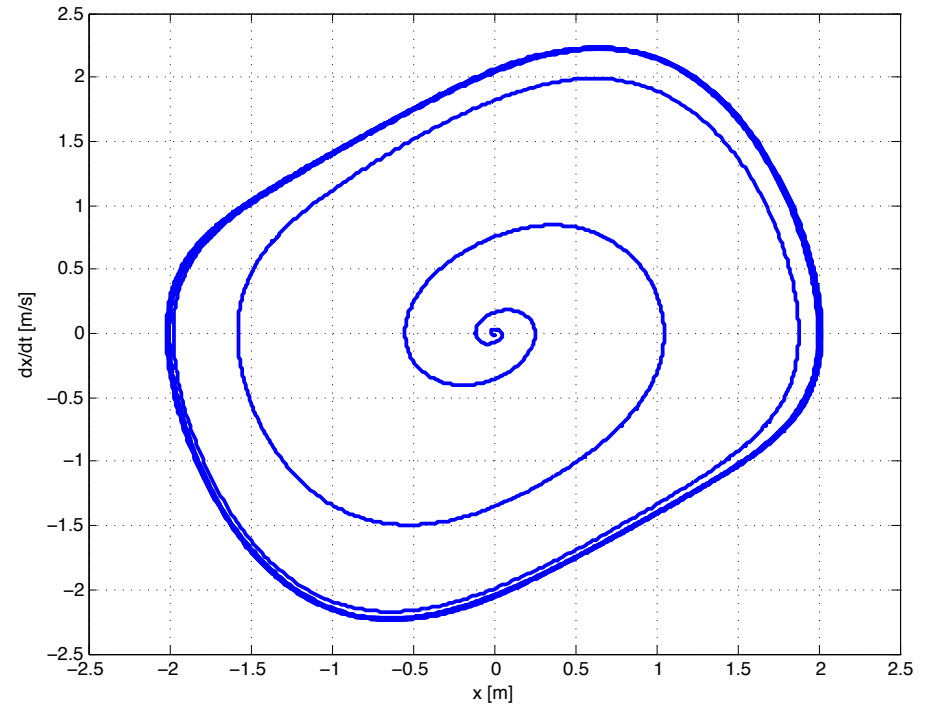
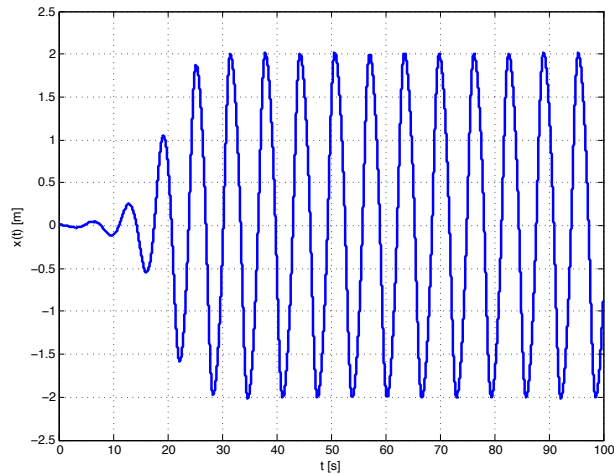


## Nonlinearity

- But now consider that inclusion of a nonlinear term could give rise to new types of behavior (e.g., limit cycles)

van der Pol oscillator

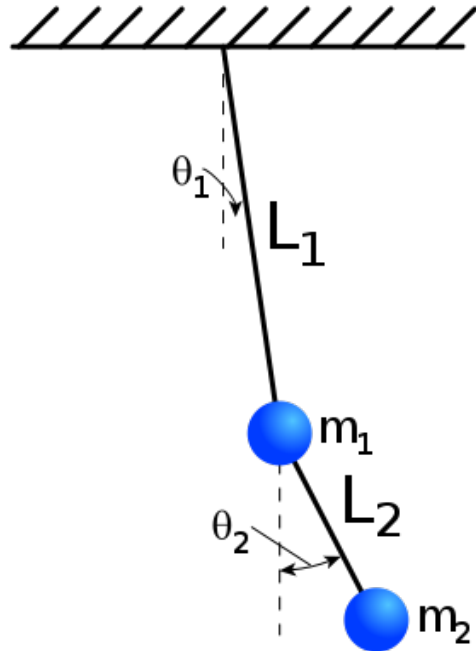
$$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$



→ We will see that a seemingly simple nonlinearity can give rise to highly complex (e.g., fractal) behavior

## Double pendulum

- Classic example of a relatively simple mechanical system, yet a nonlinear one that exhibits strikingly complex (e.g., chaotic) dynamics



<http://video.mit.edu/watch/double-pendulum-6392/>



