<u>**Computational Methods**</u> (PHYS 2030)

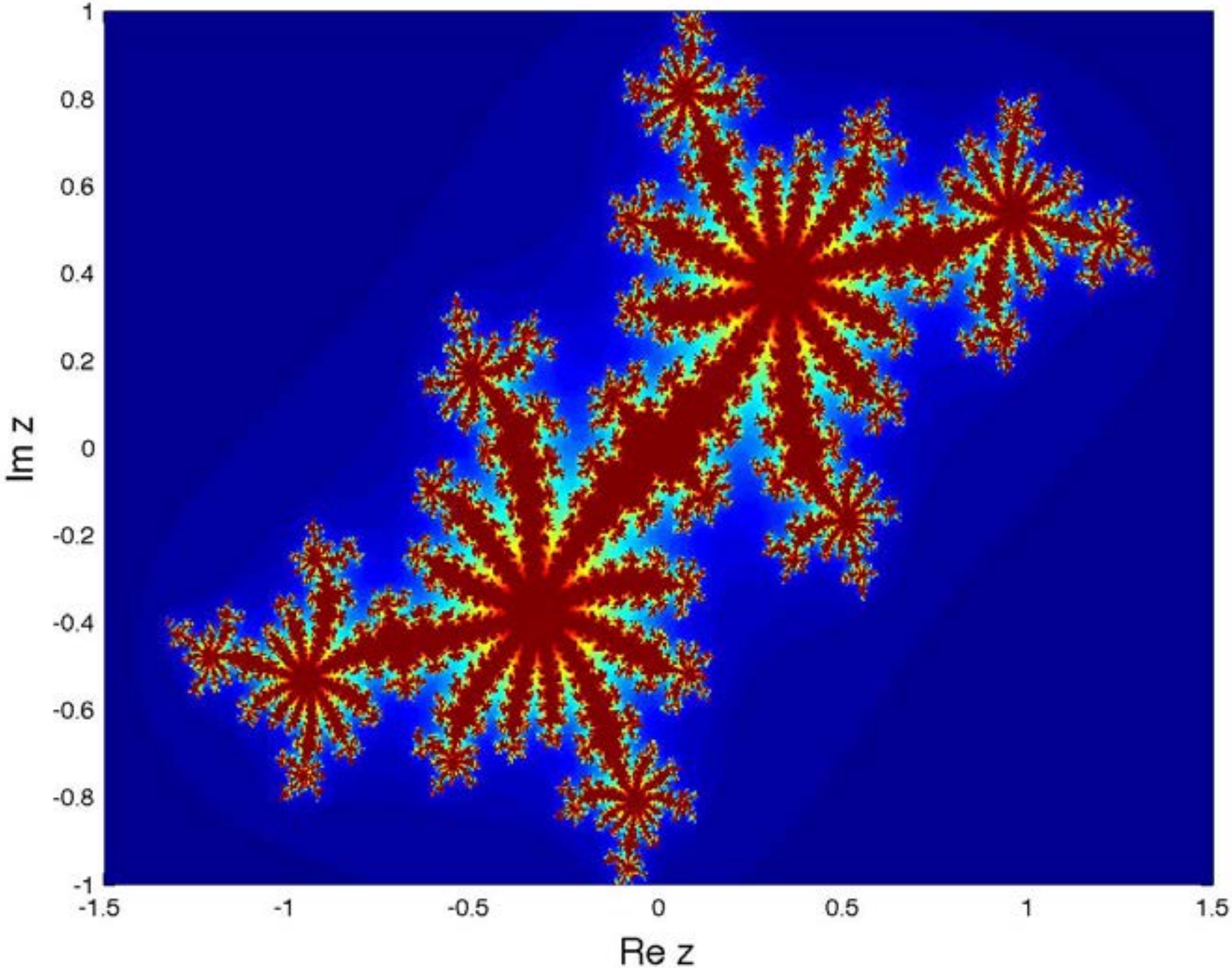
Instructors: Prof. Christopher Bergevin (cberge@yorku.ca)

Schedule: Lecture: MWF 11:30-12:30 (CLH M)

Website: http://www.yorku.ca/cberge/2030W2018.html

York University
Winter 2018
Lecture 19

EXfractal2.m



Fractal dimension

- How to quantitatively characterize fractals? (aside from drawing pretty pictures)

→ 'Fractal dimension'

Basic idea: A fractal is a curve that is something 'more than' a line, but 'less than' an area

Consider a line segment. If we divide it into N identical pieces, then each piece is described by a scale factor $r = 1/N$. Now consider a square: if it is divided into N identical pieces, each piece has linear dimensions $r = 1/\sqrt{N}$. And for a cube, $r = 1/\sqrt[3]{N}$. So, apparently, we have

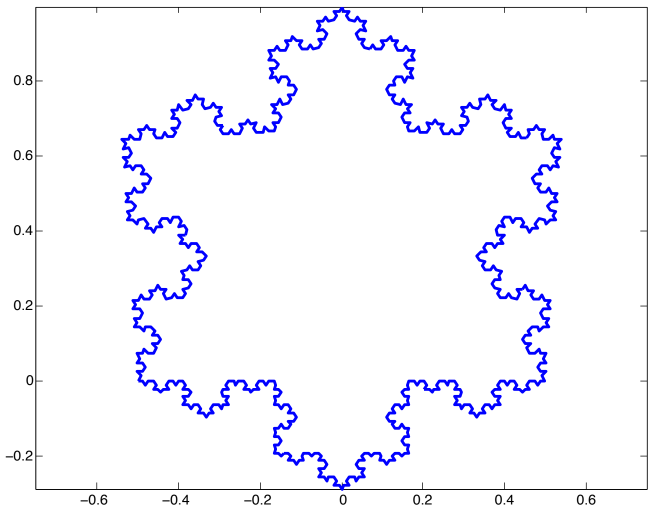
$$r = \frac{1}{\sqrt[D]{N}},$$

$$D = \frac{\log N}{\log(1/r)}$$

Fractal dimension

Devries (1994)

Fractal dimension



- Coming back to the von Koch curve:

$$F++F++F++$$

To produce a new figure, we follow two rules: first, add a T to the beginning of the instruction list; and second, replace F by a new set of instructions:

$$F \to F-F++F-F.$$

$$D = \frac{\log N}{\log(1/r)}$$

→ Each line segment is divided into four pieces (N=4) and scaled down by a factor of three (r =1/3)

$$D = \frac{\log 4}{\log 3} \approx 1.2618\ldots$$

Something 'more than' a line, but 'less than' an area

- In some sense, this is a form of a scaling law, as it tells you how something varies depending upon the scale at which you look at it (you'll see plenty more scaling laws in future physics courses)

Devries (1994)

Fractal dimension

Reports

How Long Is the Coast of Britain?

Statistical Self-Similarity and Fractional Dimension

Abstract. *Geographical curves are so involved in their detail that their lengths are often infinite or, rather, undefinable. However, many are statistically "self-similar," meaning that each portion can be considered a reduced-scale image of the whole. In that case, the degree of complication can be described by a quantity* D *that has many properties of a "dimension," though it is fractional; that is, it exceeds the value unity associated with the ordinary, rectifiable, curves.*

Seacoast shapes are examples of highly involved curves such that each of their portion can—in a statistical sense—be considered a reduced-scale image of the whole. This property will be referred to as "statistical self-similarity." To speak of a length for such figures is usually meaningless. Similarly (*1*), "the left bank of the Vistula, when measured with increased precision, would furnish lengths ten, hundred or even thousand times as great as the length read off the school map." More generally, geographical curves can be considered as superpositions of features of widely scattered characteristic size; as ever finer features are taken account of, the measured total length increases, and there is usually no clearcut gap between the realm of geography and details with which geography need not be concerned.

Quantities other than length are thus needed to discriminate between various degrees of complication for a geographical curve. When a curve is self-similar, it is characterized by an exponent of similarity, D, which possesses many properties of a dimension, though it is usually a fraction greater than the dimension 1 commonly attributed to curves. We shall reexamine in this light some empirical observations by Richardson (*2*). I propose to interpret them as implying, for example, that the dimension of the west coast of Great Britain is $D = 1.25$. Thus, the so far esoteric concept of "random figure of fractional dimension" is shown to have simple and concrete applications and great usefulness.

Self-similarity methods are a potent tool in the study of chance phenomena, including geostatistics, as well as economics (*3*) and physics (*4*). In fact, many noises have dimensions D contained between 0 and 1, so that the scientist ought to consider dimension as a continuous quantity ranging from 0 to infinity.

Returning to the claim made in the first paragraph, let us review the methods used when attempting to measure the length of a seacoast. Since a geographer is unconcerned with minute details, he may choose a positive scale G as a lower limit to the length of geographically meaningful features. Then, to evaluate the length of a coast between two of its points A and B, he may draw the shortest inland curve joining A and B while staying within a distance G of the sea. Alternatively, he may draw the shortest line made of straight segments of length at most G, whose vertices are points of the coast which include A and B. There are many other possible definitions. In practice, of course, one must be content with approximations to shortest paths. We shall suppose that measurements are made by walking a pair of dividers along a map so as to count the number of equal sides of length G of an open polygon whose corners lie on the curve. If G is small enough, it does not matter whether one starts from A or B. Thus, one obtains an estimate of the length to be called $L(G)$.

Unfortunately, geographers will disagree about the value of G, while $L(G)$ depends greatly upon G. Consequently, it is necessary to know $L(G)$ for several values of G. Better still, it would be nice to have an analytic formula linking $L(G)$ with G. Such a formula, of an entirely empirical character, was proposed by Lewis F. Richardson (*2*) but unfortunately it attracted no attention. The formula is $L(G) = M\,G^{1-D}$, where M is a positive constant and D is a constant at least equal to unity. This

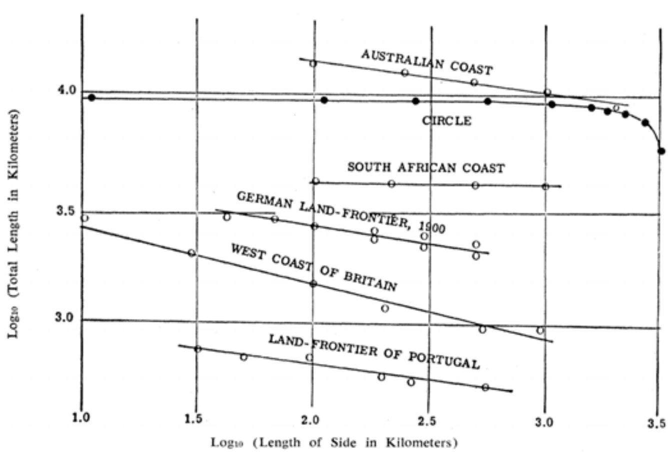


Fig. 1. Richardson's data concerning measurements of geographical curves by way of polygons which have equal sides and have their corners on the curve. For the circle, the total length tends to a limit as the side goes to zero. In all other cases, it increases as the side becomes shorter, the slope of the doubly logarithmic graph being in absolute value equal to D–1. (Reproduced from *2*, Fig. 17, by permission.)

Fractal dimension

wikipedia (fractal dimension)

Bifurcations

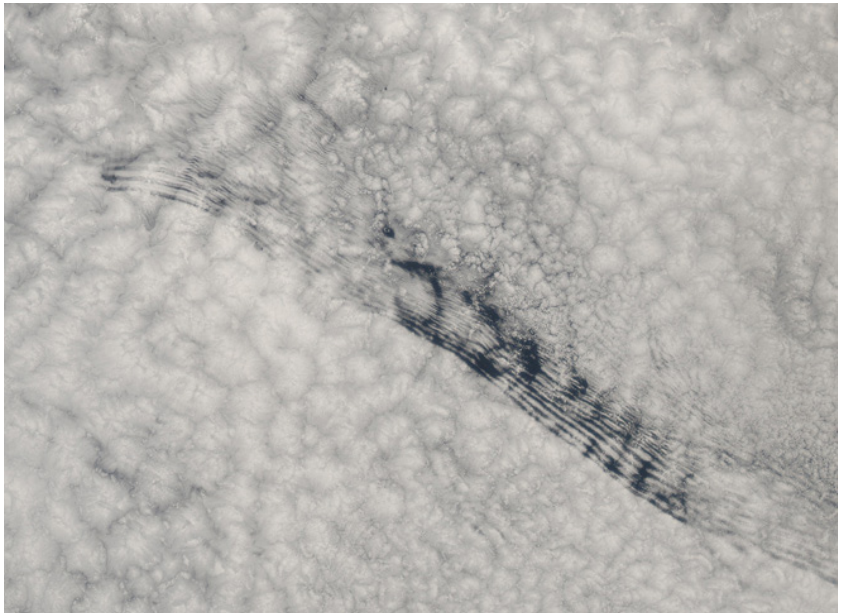
Bifurcations

Bifurcations

Fractals in nature

Leaves & plants

Clouds



Lightning

Mountains, coastlines, & river deltas

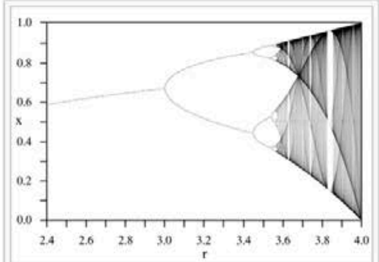
http://www.wired.com/2010/09/fractal-patterns-in-nature/

Bifurcations

- In the most general sense, a ‘bifurcation’ describes how something ‘splits’
- In dynamical systems theory, bifurcation analysis is a powerful means to study nonlinear systems

Minimum complexity of a chaotic system [edit]

Discrete chaotic systems, such as the logistic map, can exhibit strange attractors whatever their dimensionality. In contrast, for continuous dynamical systems, the Poincaré–Bendixson theorem shows that a strange attractor can only arise in three or more dimensions. Finite-dimensional linear systems are never chaotic; for a dynamical system to display chaotic behavior, it has to be either nonlinear or infinite-dimensional.



Bifurcation diagram of the logistic map $x \to r\,x\,(1-x)$. Each vertical slice shows the attractor for a specific value of r. The diagram displays period-doubling as r increases, eventually producing chaos.

The Poincaré–Bendixson theorem states that a two-dimensional differential equation has very regular behavior. The Lorenz attractor discussed above is generated by a system of three differential equations such as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma y - \sigma x,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \rho x - xz - y,$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

- Minimum complexity?
- Bifurcation diagram?

where x, y, and z make up the system state, t is time, and σ, ρ, β are the system parameters. Five of the terms on the right hand side are linear, while two are quadratic; a total of seven terms. Another well-known chaotic attractor is generated by the Rossler equations which have only one nonlinear term out of seven. Sprott [20] found a three-dimensional system with just five terms, that had only one nonlinear term, which exhibits chaos for certain parameter values. Zhang and Heidel [21][22] showed that, at least for dissipative and conservative quadratic systems, three-dimensional quadratic systems with only three or four terms on the right-hand side cannot exhibit chaotic behavior. The reason is, simply put, that solutions to such systems are asymptotic to a two-dimensional surface and therefore solutions are well behaved.

While the Poincaré–Bendixson theorem shows that a continuous dynamical system on the Euclidean plane cannot be chaotic, two-dimensional continuous systems with non-Euclidean geometry can exhibit chaotic behavior.[23] Perhaps surprisingly, chaos may occur also in linear systems, provided they are infinite dimensional.[24] A theory of linear chaos is being developed in a branch of mathematical analysis known as functional analysis.

wikipedia (~1/3 of the way down for ‘chaos theory’)

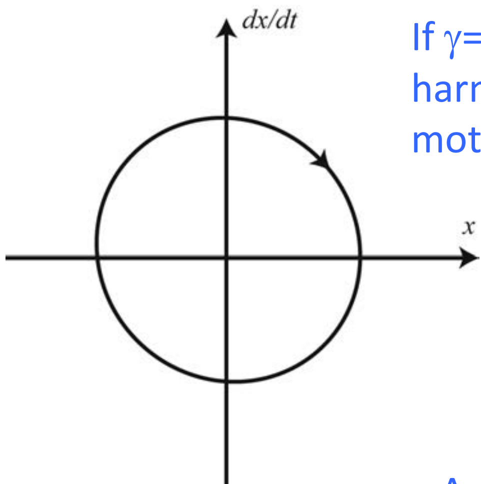
Bifurcations

- In the context of dynamical systems, consider bifurcation analysis as a means to assess how the overall ('qualitative') behavior of a system depends upon the parameters (i.e., the 'constants')

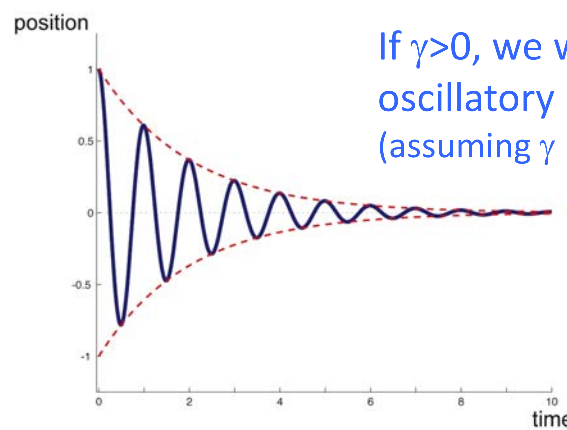
→ Consider the damped, undriven harmonic oscillator with a nonzero initial condition

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = 0$$

$$\text{e.g.,} \quad x(0) = 0, \quad \dot{x} = 1$$



If γ=0, we will have simple harmonic (i.e., periodic) motion



If γ>0, we will have damped oscillatory motion (assuming γ is not too big!)

And if γ<0, then what?

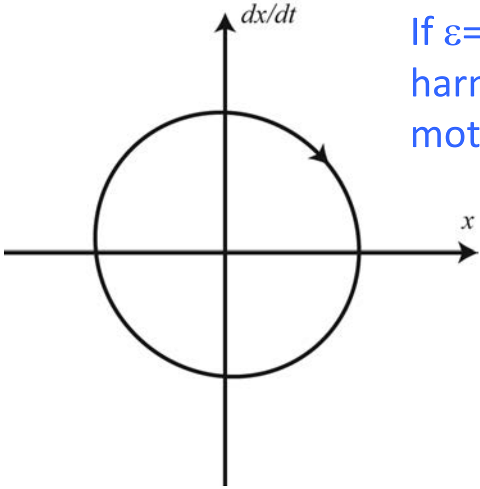
- Clearly there is a change in the behavior of the system about γ=0

→ We call this a bifurcation

ex. van der Pol oscillator

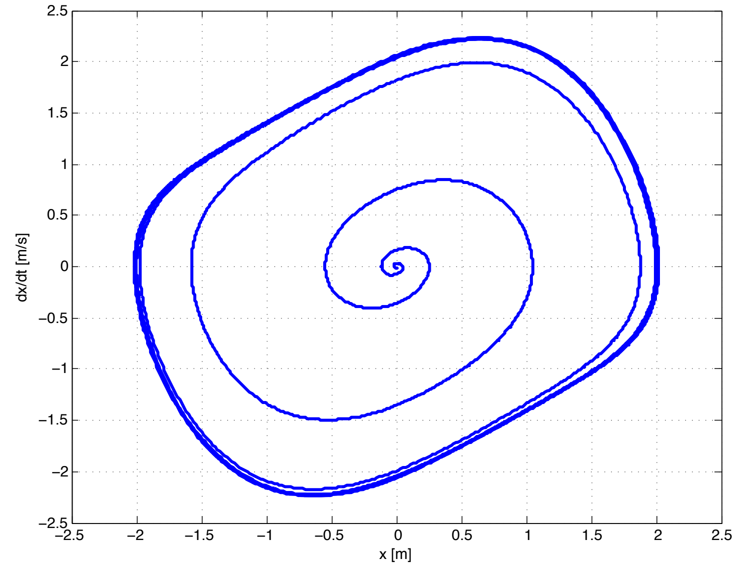
- So what about a nonlinear oscillator?

$$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$



If ε=0, we will have simple harmonic (i.e., periodic) motion

If ε>0, we will have stable limit cycle



- Physically, a very different type of behavior emerges when ε goes from 0 to a positive value (this is actually called a *supercritical Hopf bifurcation*)

Hopf bifurcation - "... a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane. Under reasonably generic assumptions about the dynamical system, we can expect to see a small-amplitude limit cycle branching from the fixed point."

wikipedia (Hopf bifurcation)

Ex. Critical oscillators in hearing?

J Neurophysiol 104: 1219–1229, 2010.
First published June 10, 2010; doi:10.1152/jn.00437.2010.

Review

A Critique of the Critical Cochlea: Hopf—a Bifurcation—Is Better Than None

A. J. Hudspeth,[1] Frank Jülicher,[2] and Pascal Martin[3]
[1]*Howard Hughes Medical Institute and Laboratory of Sensory Neuroscience, The Rockefeller University, New York, New York;* [2]*Max-Planck-Institut für Physik komplexer Systeme, Dresden, Germany; and* [3]*Laboratoire Physico-Chimie Curie, Centre National de la Recherche Scientifique, Institut Curie, Université Pierre et Marie Curie, Paris, France*

Hudspeth AJ, Jülicher F, Martin P. A critique of the critical cochlea: Hopf—a bifurcation—is better than none. *J Neurophysiol* 104: 1219–1229, 2010. First published June 10, 2010; doi:10.1152/jn.00437.2010. The sense of hearing achieves its striking sensitivity, frequency selectivity, and dynamic range through an active process mediated by the inner ear's mechanoreceptive hair cells. Although the active process renders hearing highly nonlinear and produces a wealth of complex behaviors, these various characteristics may be understood as consequences of a simple phenomenon: the Hopf bifurcation. Any critical oscillator operating near this dynamic instability manifests the properties demonstrated for hearing: amplification with a specific form of compressive nonlinearity and frequency tuning whose sharpness depends on the degree of amplification. Critical oscillation also explains spontaneous otoacoustic emissions as well as the spectrum and level dependence of the ear's distortion products. Although this has not been realized, several valuable theories of cochlear function have achieved their success by incorporating critical oscillators.

Basic idea: Sensory cells of your ear act something akin to van der Pol oscillators, thereby allowing them to act like amplifiers

Ex. Critical oscillators in hearing?

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$$\dot{Z} = -(C - C_C - i\omega_C)Z - b|Z|^2 Z + \frac{F}{\Lambda}$$

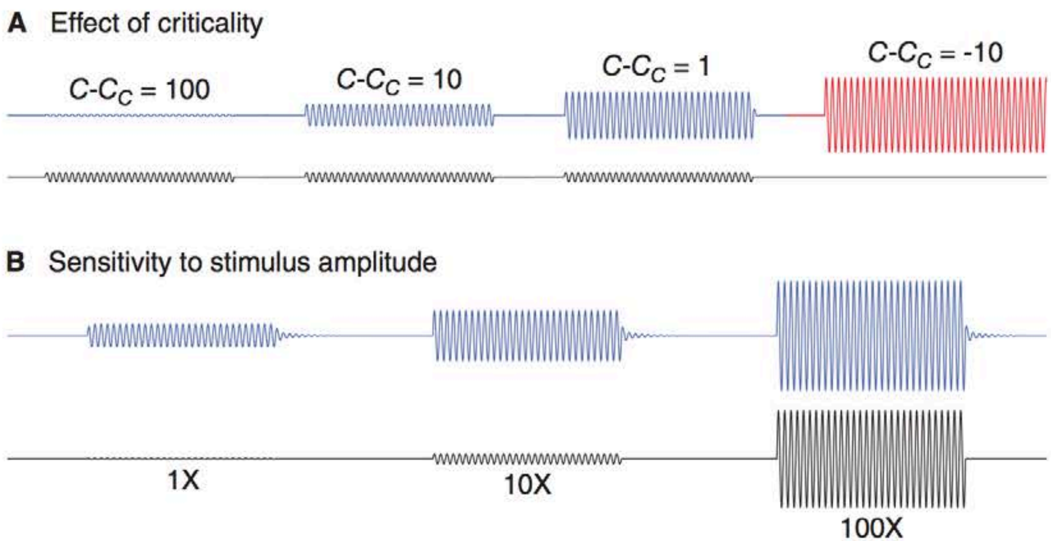

FIG. 1. The behavior of a critical oscillator. *A*: an oscillator whose response is described by *Eq. 1* produces sinusoidal responses (*top traces*) when driven at its characteristic frequency with pulses of sinusoidal stimuli (*bottom traces*). As the difference between the value of the control parameter and the critical value, $C - C_C$, declines from 100 to 10 and then to 1, the response to identical stimuli grows substantially. Finally, when $C - C_C = -10$, the system becomes unstable and undergoes limit-cycle oscillation in the absence of stimulation. *B*: when the oscillator operates near criticality, here with $C - C_C = 0.1$ in each panel, its response exhibits compressive nonlinearity. As the stimulus grows by successive factors of 10, the responses increase by factors of only $10^{1/3}$ or about 2.2.

$$z_{n+1} = z_n^2 + c \qquad z, c \in \mathbb{C}$$

$\leftarrow \rightarrow$

$$x_{n+1} = x_n^2 - y_n^2 + \mathrm{Re}(c)$$
$$y_{n+1} = 2x_n y_n + \mathrm{Im}(c)$$

$$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$

Remember: A single complex differential equation is (more or less) equivalent to a system of two first order real-valued equations

Somewhat easier to think of in terms of a 2nd order (real) ODE, where ε is the ‘control parameter’

Ex. Critical oscillators in hearing?

- In science, not everyone agrees… (this is a good thing)

CELLULAR COOPERATION IN COCHLEAR MECHANICS

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Two contrasting views of cochlear mechanics are compared with each other, and with experiment. The first posits that all qualitative features of the nonlinear cochlear response are those of a simple dynamical system poised at a Hopf bifurcation, the second argues that the cochlear response must be found with 3-D simulations. Hopf bifurcations are explained, and their consequences explored.

"The saccular hair cells of the bullfrog, and the basilar membrane of the cochlea, both have nonlinear responses. In common with many nonlinear systems, they exhibit compression and potential instability. The functional form of that compression is quite different for these two systems. Therefore, whatever resemblance they may appear to have is superficial, and the responses of these two systems have little in common, even qualitatively. Nonlinear aspects of hearing do not arise from a cochlea poised near a Hopf bifurcation. The cochlea is poised elsewhere. More generally, the properties of generic dynamical systems are not expected to be applicable to hearing, since these systems exhibit relevant properties only asymptotically, while hearing is concerned primarily with transient sounds."

from *Biophysics of the Cochlea – From Molecules to Models* (2003)

Period doubling

- Let's return to this picture from before and the notion of 'minimum complexity'

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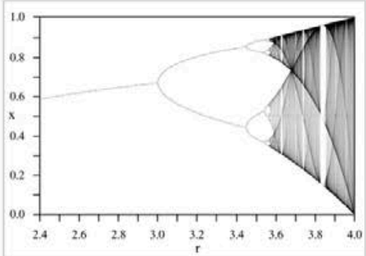

Bifurcation diagram of the logistic map $x \rightarrow r\,x\,(1 - x)$. Each vertical slice shows the attractor for a specific value of r. The diagram displays period-doubling as r increases, eventually producing chaos.

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- We'll explore a relatively simple nonlinearity to see what emerges: the logistic map

Period doubling

- Let us return to the (discrete version of the) nonlinear logistic equation we've previously examined for nonlinear population growth

Logistic eqn. (continuous)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

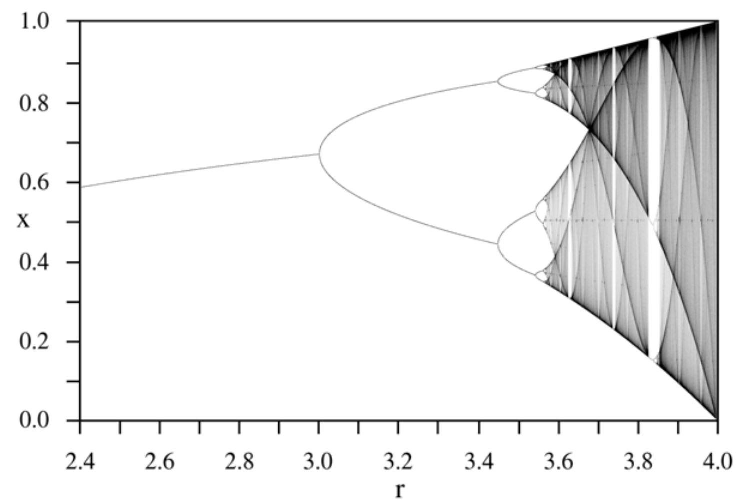
Logistic eqn. (discrete)

$$P_{n+1} = kP_n(1 - P_n/L)$$

- We'll simplify slightly (but keep real-valued): $x_{n+1} = x_n r(1 - x_n)$

- Relatively innocuous equation, right?

→ Our goal is to numerically perform a bifurcation analysis (with respect to r) and observe how 'period doubling' emerges, commonly pointed towards as a characteristic of *chaos*

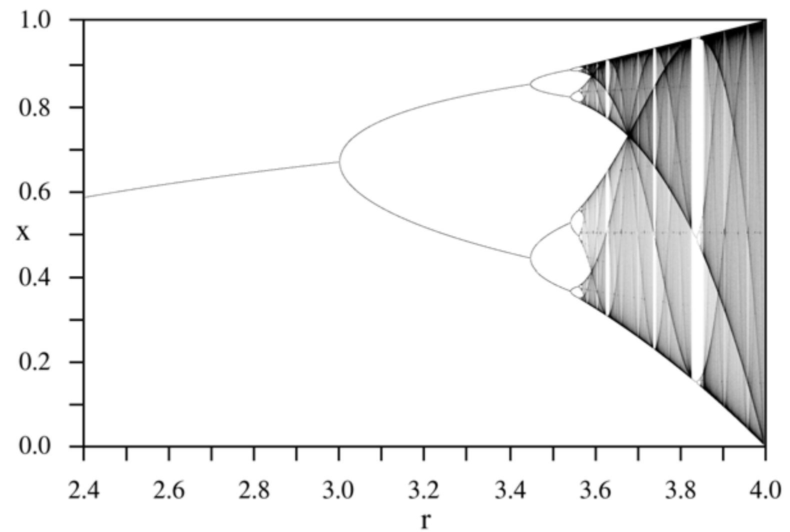


wikipedia (chaos theory)

Period doubling

- Slight modification to our procedure earlier for producing fractals

$$x_{n+1} = x_n r(1 - x_n)$$



1. Choose a range of value for r and step through a handful (Nr) of different values. For each, have a starting x_0= const. (e.g., 0.1)

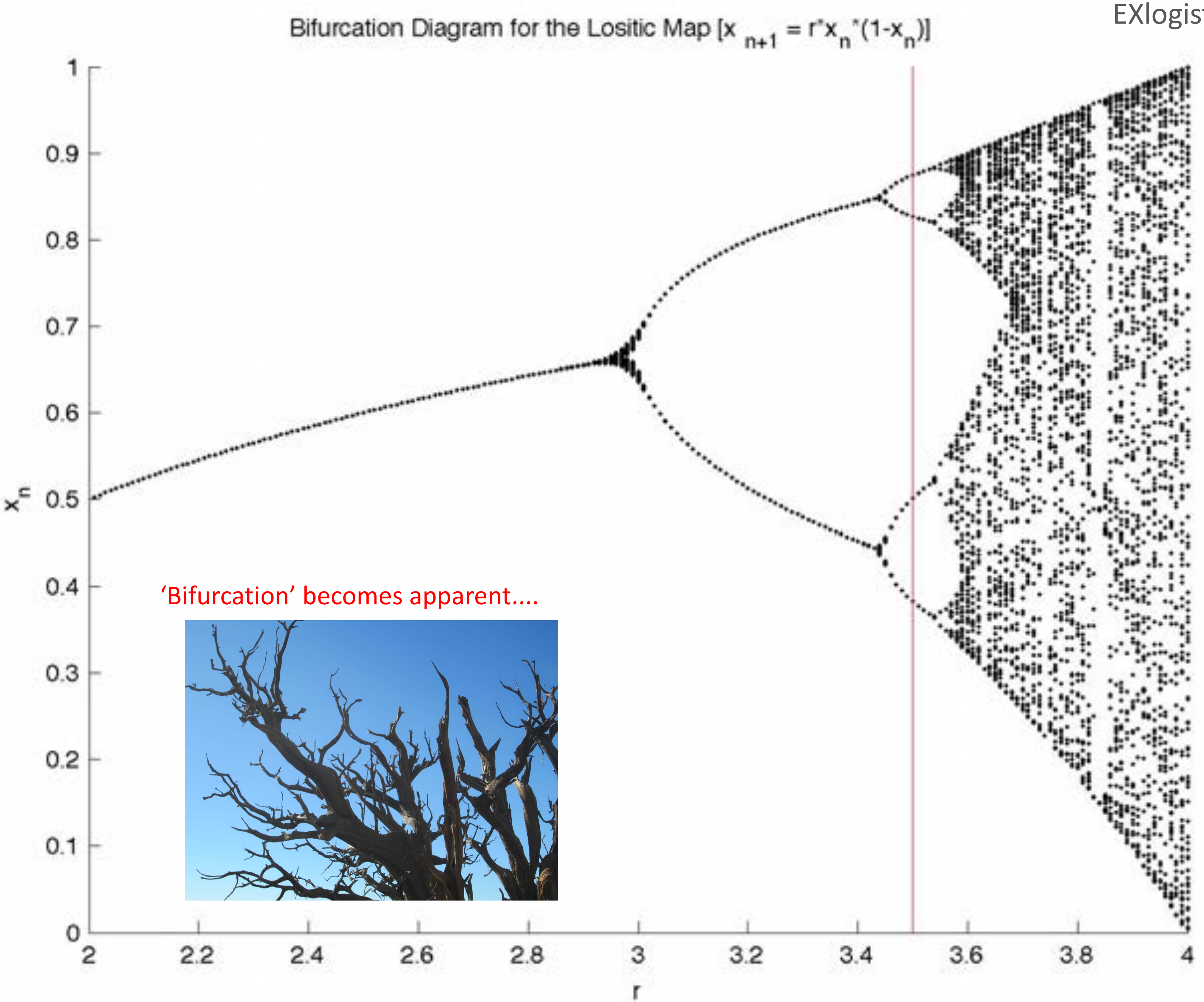
2. We then 'iterate' the logistic map forward (say, N times) for that r. Keep track of all values as the map is iterated.

3. After some number of iterations (Nsettl) to allow for 'settling', simply plot all the values (Nit) of x_n for that r. Rinse and repeat....

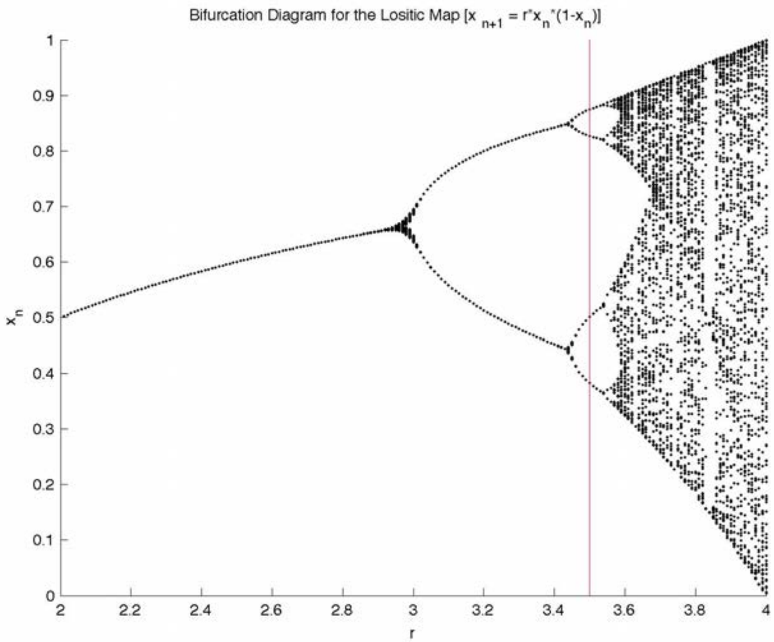
→ Again a relatively simple recipe, but we'll see what it results in....

```
clear;  figure(1); clf; hold on;
% ---------------
% User inputs
range= [2 4];   % min and max values to compute bifurcation diagram over [2 4]
Nr= 200;      % # of steps over range [100]
x0= 0.1;     % starting x value [0.1]
Nsettl= 50; % # of runs allowed for 'settling' [50]
Nit= 100;   % # of iterations to plot for a given value of r [200]
rPlot= 3.5; % for 'timecourse' plot, specify associated r value (must be inside range!)
% ---------------
rmin= range(1); rmax= range(2);
% loop through each r value
for nn=1:Nr
    r(nn)= rmin + nn*(rmax-rmin)/Nr;    % update r
    x= x0;  % reset to IC
    xS(1)= x;   % store first point
    indx=2; % reset indexer (for 2nd iterate)
    for mm=0:Nsettl+Nit     % loop through the iterations of the map
        x= r(nn)*x*(1-x);       % deal with mapping
        xS(nn,indx)= x;         % store values
        indx= indx+1;        % update indexer
    end
    % plot points for a given iteration *past* the settling time
    plot(r(nn)*ones(Nit+1),xS(nn,Nsettl:Nsettl+Nit),'k.')
end
% ----
xlabel('r');    ylabel('x_n')
title('Bifurcation Diagram for the Lositic Map [x_{n+1} = r*x_n*(1-x_n)]')
% ----
% also plot x_n as function of n for relevant r value (as specified)
[junk indxR] = min(abs(r-rPlot));   % search for closest r value to rPlot
n= linspace(0,size(xS,2),size(xS,2));
figure(2); clf;
plot(n,xS(indxR,:),'kd-'); hold on;
xlabel('n');    ylabel('x_n');
stem(Nsettl,max(xS(indxR,:)),'r-','marker','none'); % indicate bound for 'settling'
figure(1); stem(rPlot,max(xS(:)),'r-','marker','none'); % indicate r for which 'time course' is plotted
```

EXlogisticBIF.m

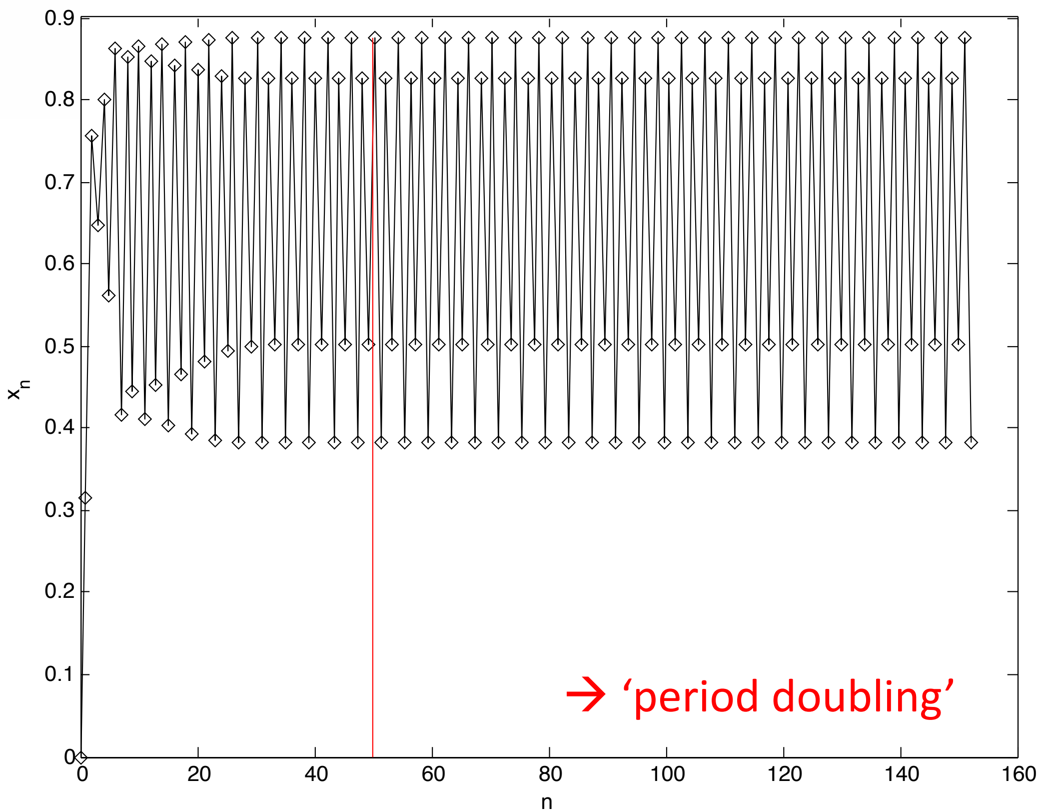


‘Bifurcation’ becomes apparent....



- After the settling period, for larger values of r, higher and higher orders of periods (i.e., oscillations between different values) emerge

→ Once r is large enough (>3.6), there are so many different 'hopping points' that the behavior looks erratic or noisy (even though there is a predictable underlying structure)



EXlogisticBIF.m

```
r = 3.8;
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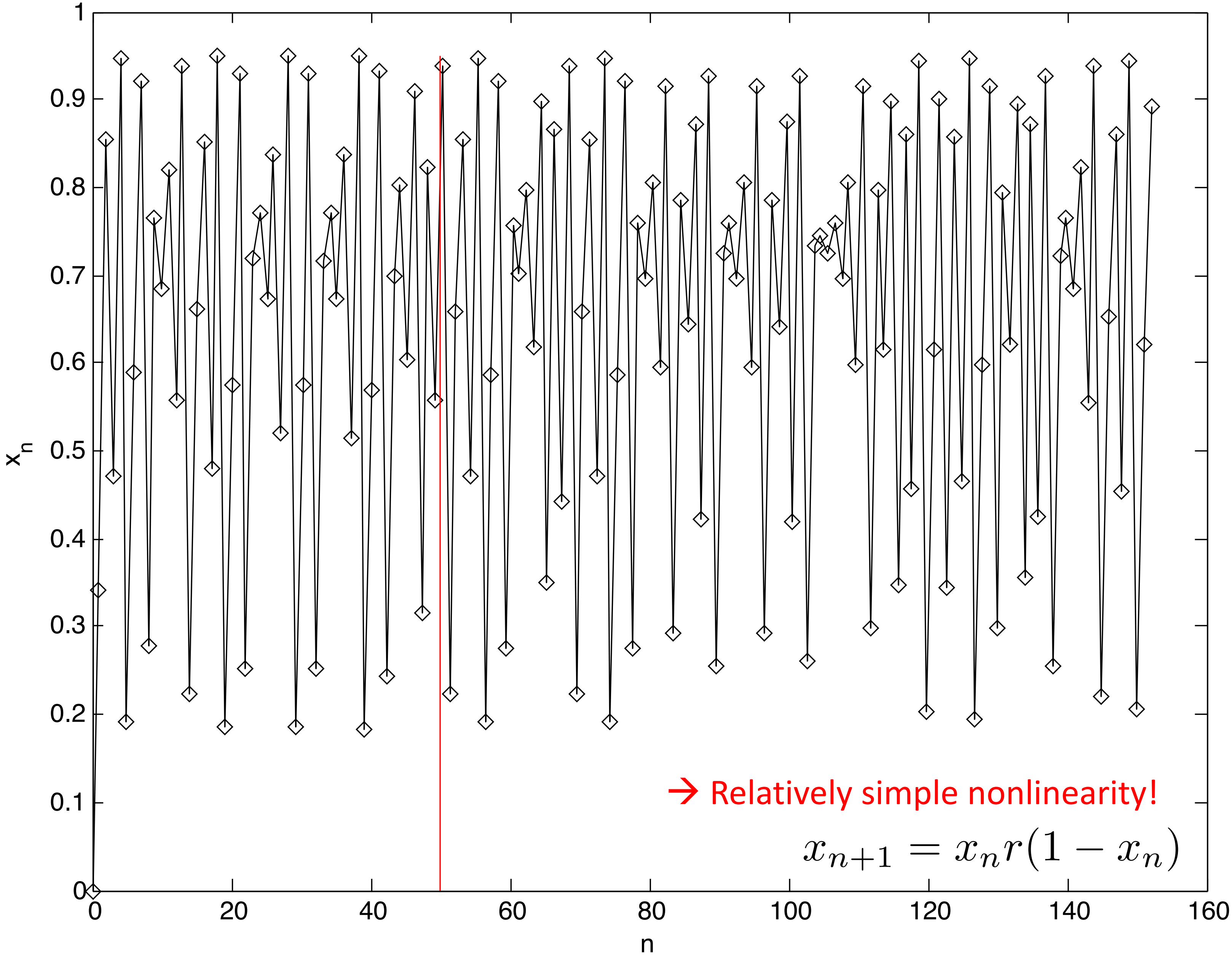


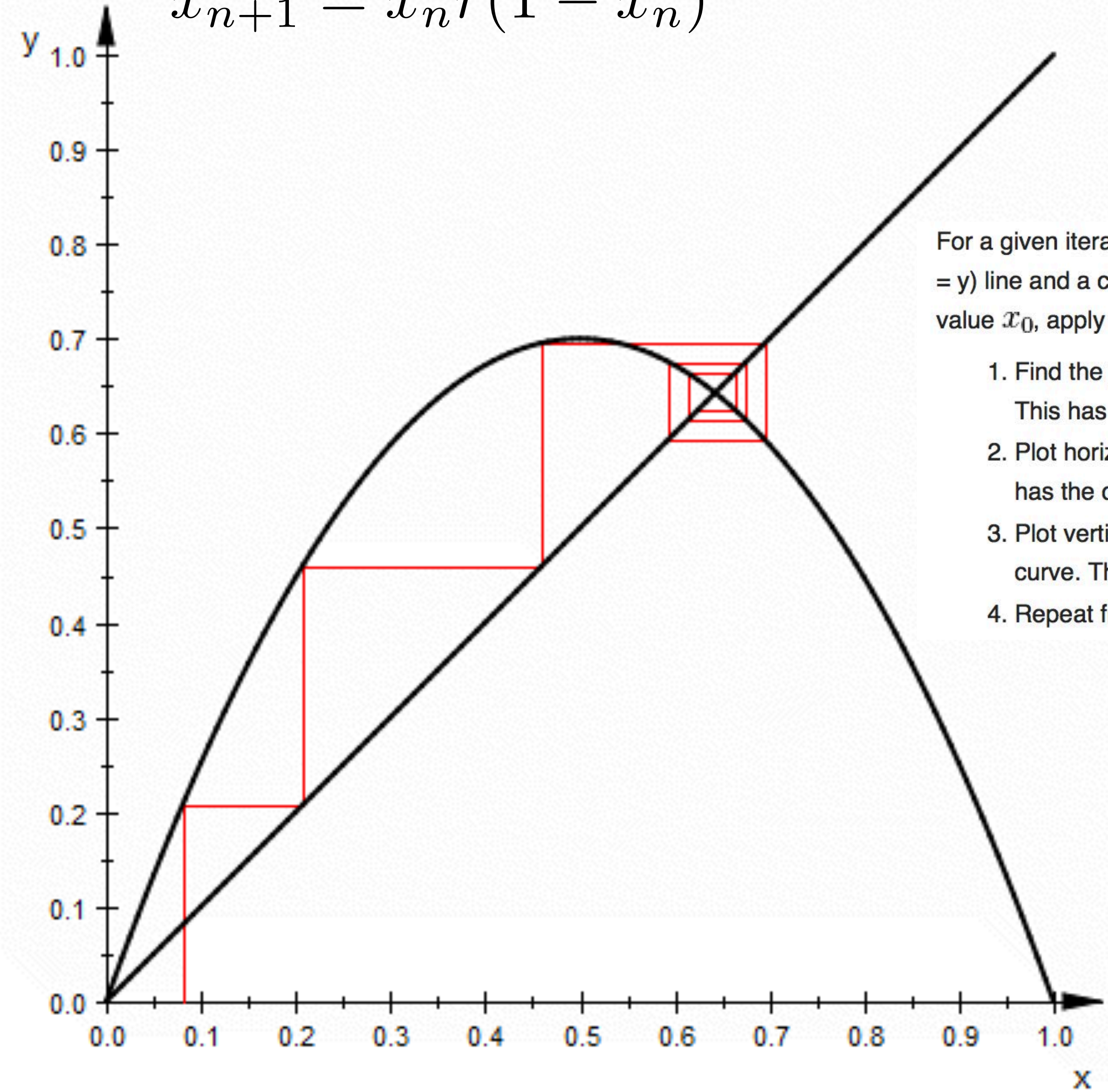
→ Relatively simple nonlinearity!

$$x_{n+1} = x_n r(1 - x_n)$$

“Cobweb plot”

- Other useful ways to visualize the ‘dynamics’....

$$x_{n+1} = x_n r(1 - x_n)$$



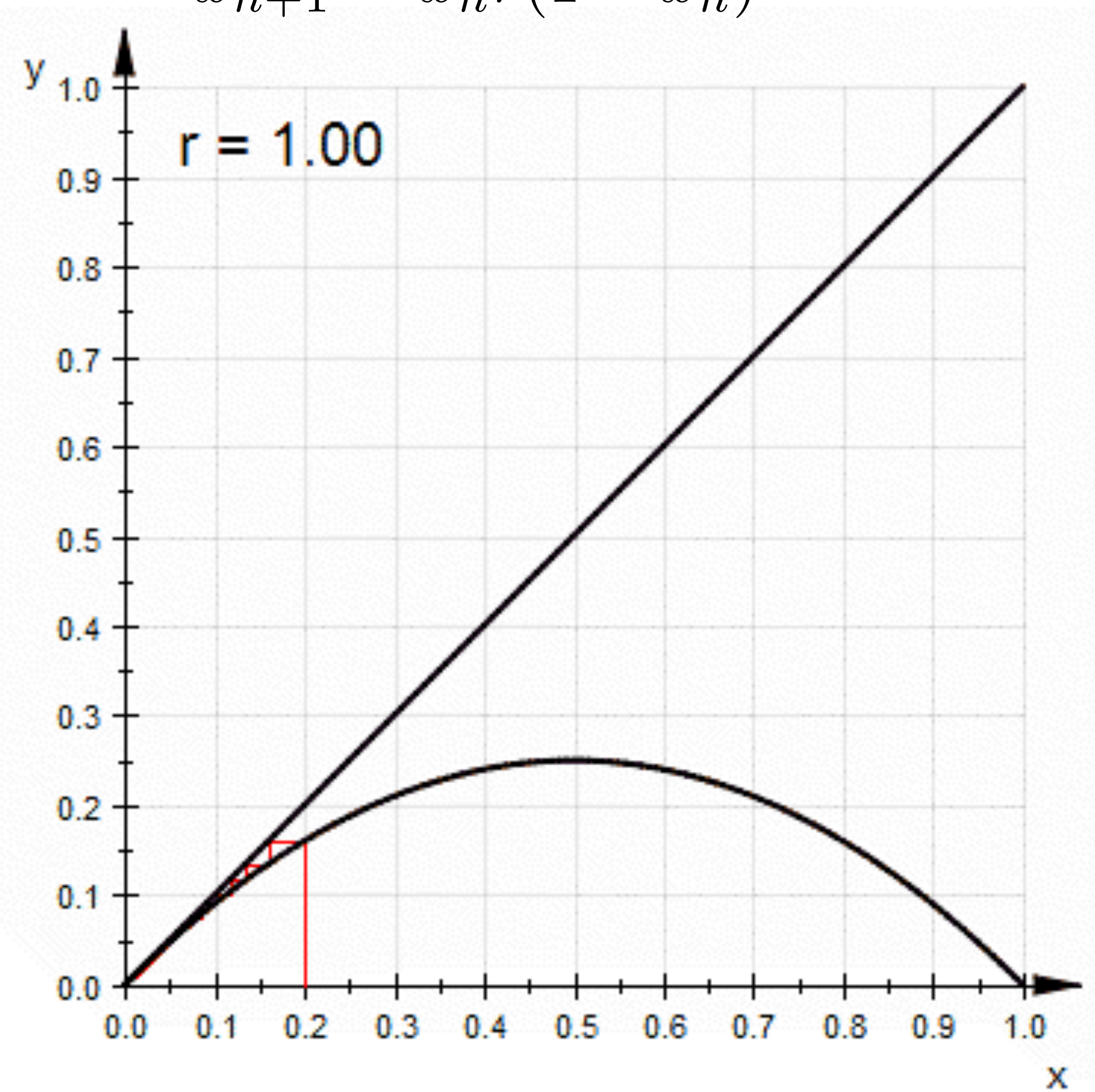
For a given iterated function *f*: **R** → **R**, the plot consists of a diagonal (x = y) line and a curve representing y = f(x). To plot the behaviour of a value x_0, apply the following steps.

1. Find the point on the function curve with an x-coordinate of x_0. This has the coordinates $(x_0, f(x_0))$.
2. Plot horizontally across from this point to the diagonal line. This has the coordinates $(f(x_0), f(x_0))$.
3. Plot vertically from the point on the diagonal to the function curve. This has the coordinates $(f(x_0), f(f(x_0)))$.
4. Repeat from step 2 as required.

wikipedia (cobweb plot)

Period doubling

$$x_{n+1} = x_n r(1 - x_n)$$



- Other useful ways to visualize the ‘dynamics’....

→ Further motivates the notion of ‘minimal complexity’.....

wikipedia (logistic map)

Nature Vol. 261 June 10 1976

review article

Simple mathematical models with very complicated dynamics

Robert M. May*

First-order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. This is an interpretive review of them.

→ According to Google Scholar, this paper has well over 5000 citations (i.e., it has had a significant 'impact')

review article

Simple mathematical models with very complicated dynamics

Robert M. May*

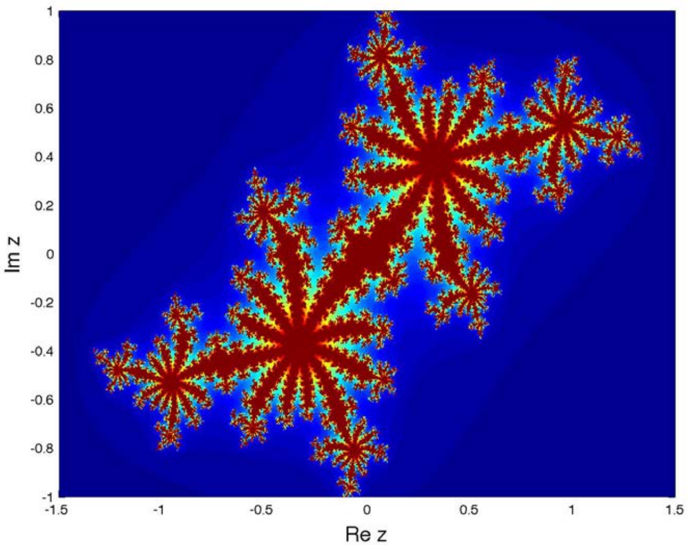
First-order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. This is an interpretive review of them.

$$X_{t+1} = aX_t(1 - X_t) \tag{3}$$

The elegant body of mathematical theory pertaining to linear systems (Fourier analysis, orthogonal functions, and so on), and its successful application to many fundamentally linear problems in the physical sciences, tends to dominate even moderately advanced University courses in mathematics and theoretical physics. The mathematical intuition so developed ill equips the student to confront the bizarre behaviour exhibited by the simplest of discrete nonlinear systems, such as equation (3). Yet such nonlinear systems are surely the rule, not the exception, outside the physical sciences.

I would therefore urge that people be introduced to, say, equation (3) early in their mathematical education. This equation can be studied phenomenologically by iterating it on a calculator, or even by hand. Its study does not involve as much conceptual sophistication as does elementary calculus. Such study would greatly enrich the student's intuition about nonlinear systems.

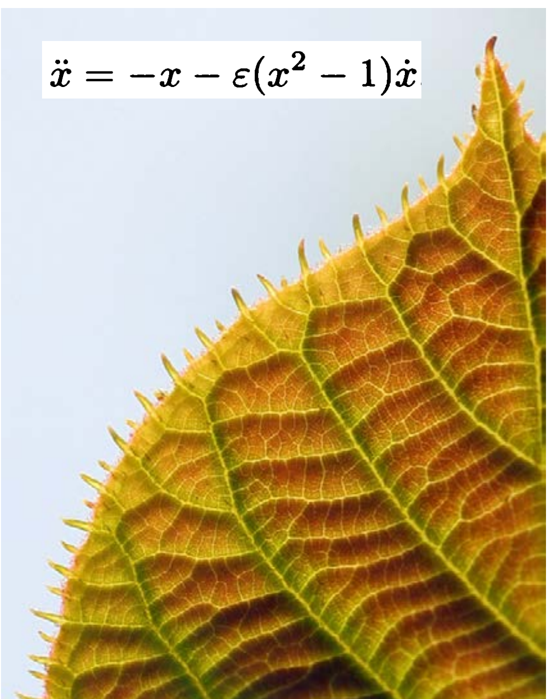
Summary



Fractals can be generated by relatively simple nonlinearities

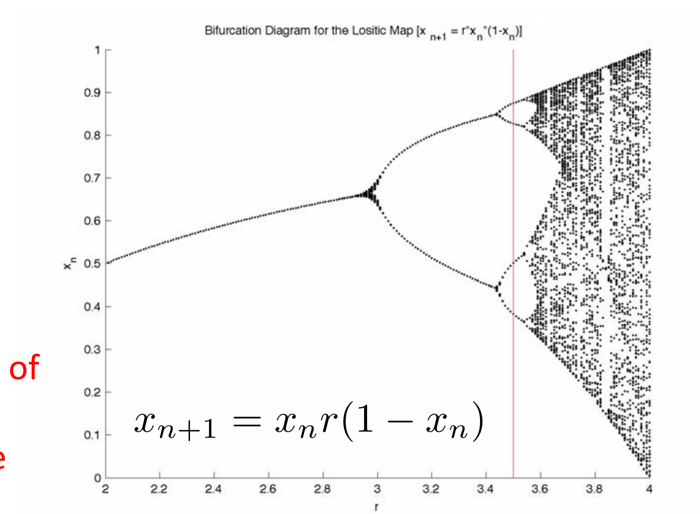
$$D = \frac{\log N}{\log(1/r)}$$

Fractal dimensions indicate 'something between a line and an area'



Bifurcations appear in a wide variety of context, from fractal geometry to dynamics of mechanical systems

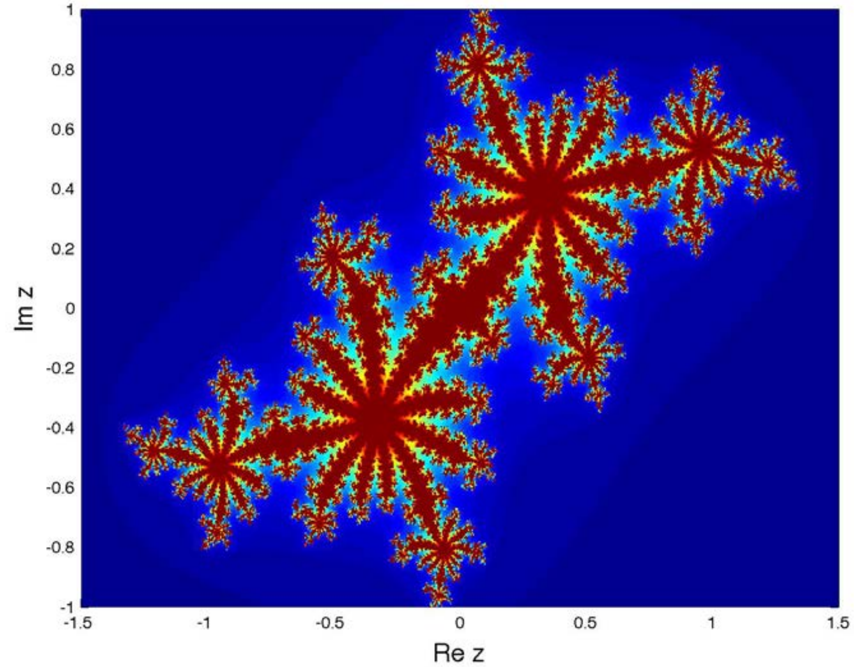
Period doubling is a type of complex behavior that arises in relatively simple nonlinear systems



Post-class exercises

- How can you modify EXfraxtal2.m so to produce different fractal shapes?

- Run the period doubling code (EXlogisticBIF.m). Look at the timecourse for different values of r. Do they match what you see on the bifurcation diagram?



- Does changing the various parameters for EXlogisticBIF.m (e.g., initial condition, settling period) have any substantial effect?

- Get a copy of May's 1976 paper and read it. What other aspects to this nonlinear system become apparent?