Computational Methods (PHYS 2030)

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Schedule: Lecture: MWF 11:30-12:30 (CLH M)

Website: http://www.yorku.ca/cberge/2030W2018.html

York University
Winter 2018
Lecture 21

Review

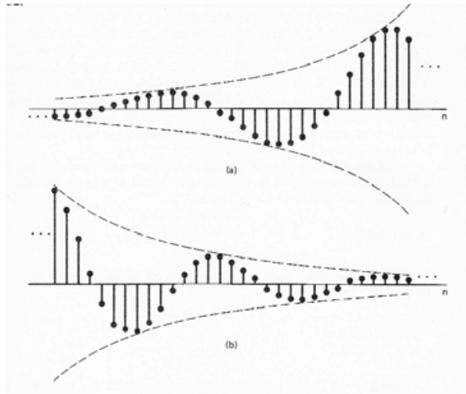
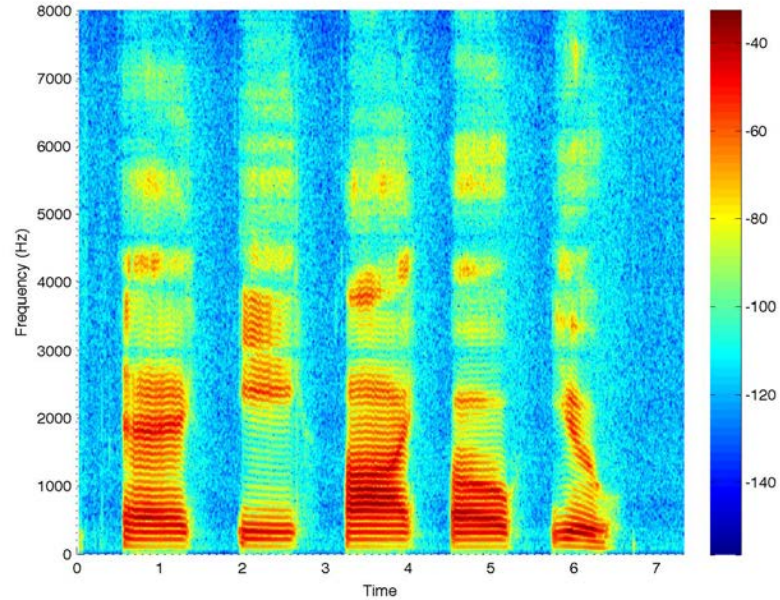
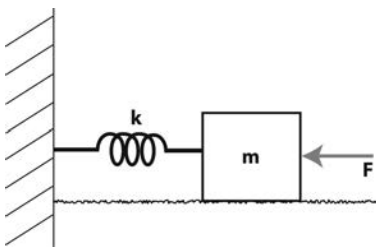
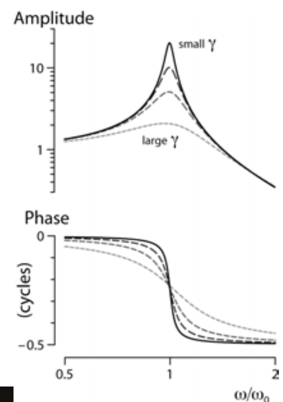
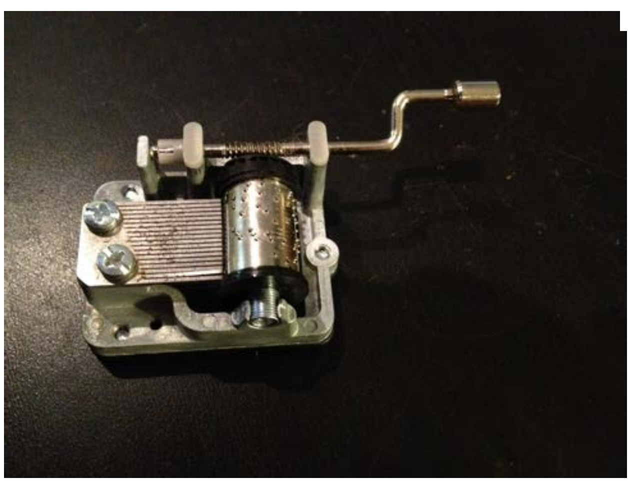


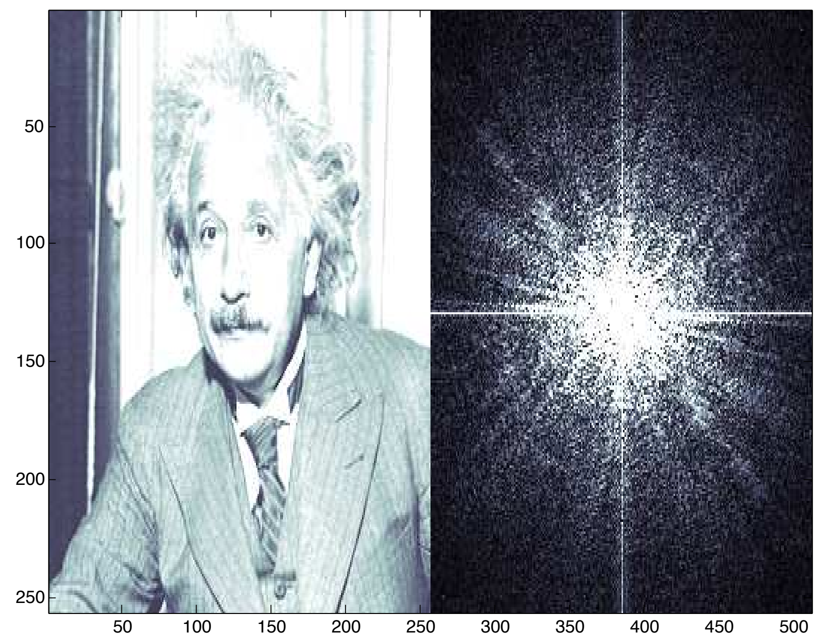
Figure 2.32 (a) Growing discrete-time sinusoidal signal; (b) decaying discrete-time sinusoid.











Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

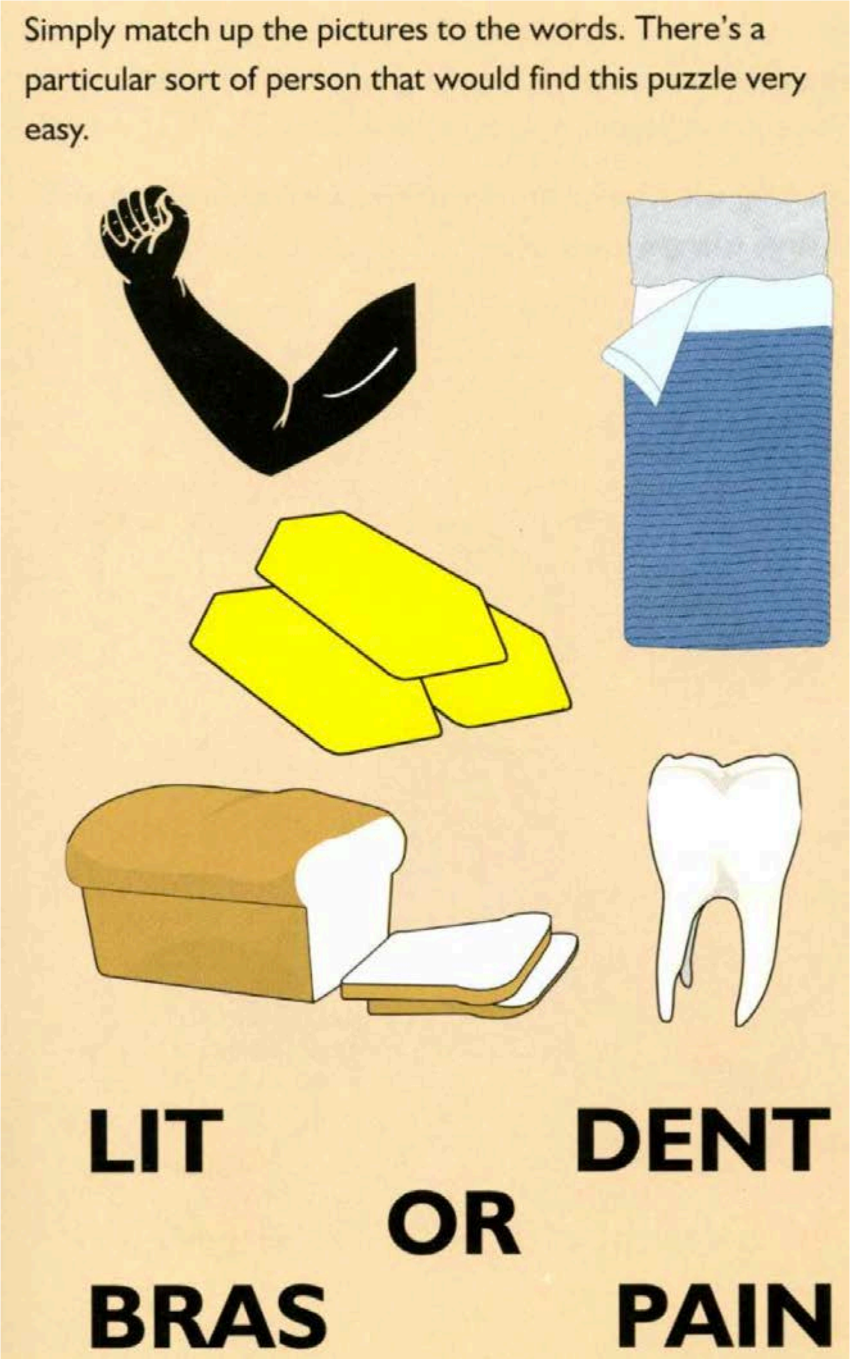
- Fourier series are useful for describing functions that:
  - *exhibit some degree of periodicity*
  - might have sharp discontinuous behavior (like sampled signals!)

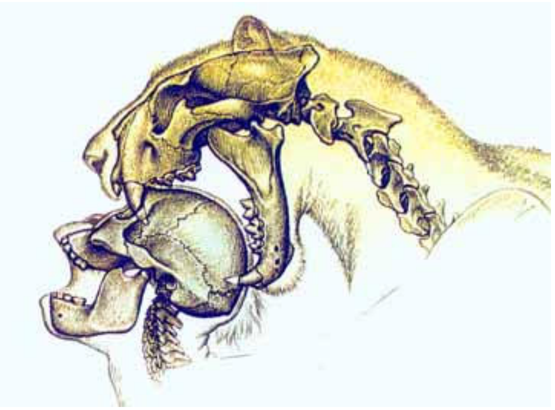
Joseph Fourier (1768-1830)

- When Fourier presented this idea to the French Academy of Science in 1812, the panel of referees (Lagrange, Laplace, and Legendre) were skeptical. They were were worried that this series representation would converge.

Devries (1994)

Basic idea: Represent ‘signal’ as a sum of sinusoids





Bed

Tooth

Gold

Arm

Bread

- Think in the context of '*information*'
- There are many different ways we can '*encode*' information
- We just might use different *bases* to describe a given piece of information

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Fourier series

Intuitive connection back to Taylor series:

$$y(x_1 + \Delta x) \approx y(x_1) + \sum_{n=1}^{N} \frac{1}{n!} \left.\frac{d^n y}{dx^n}\right|_{x_1} (\Delta x)^n. \qquad \text{(D.2)}$$

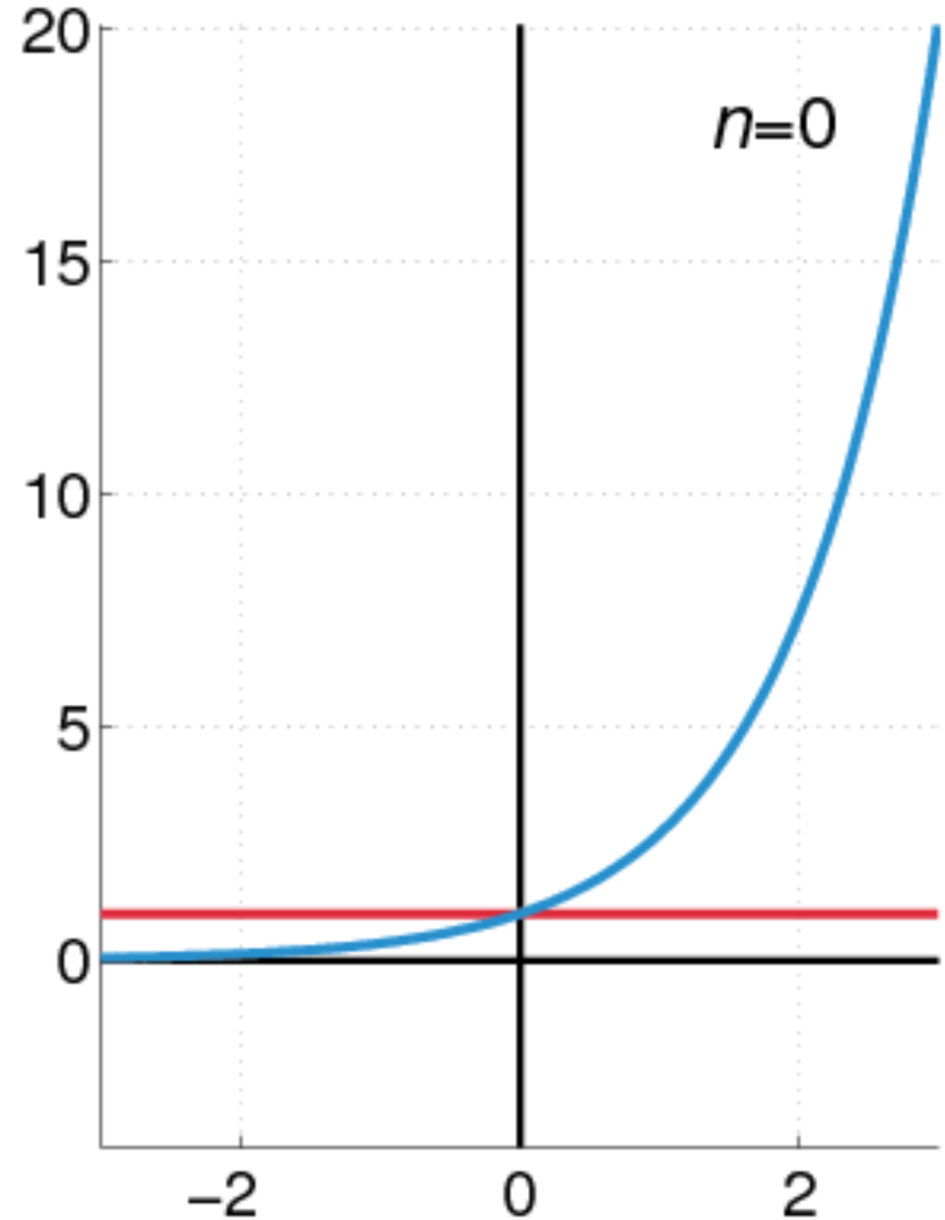
$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \cdots + \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

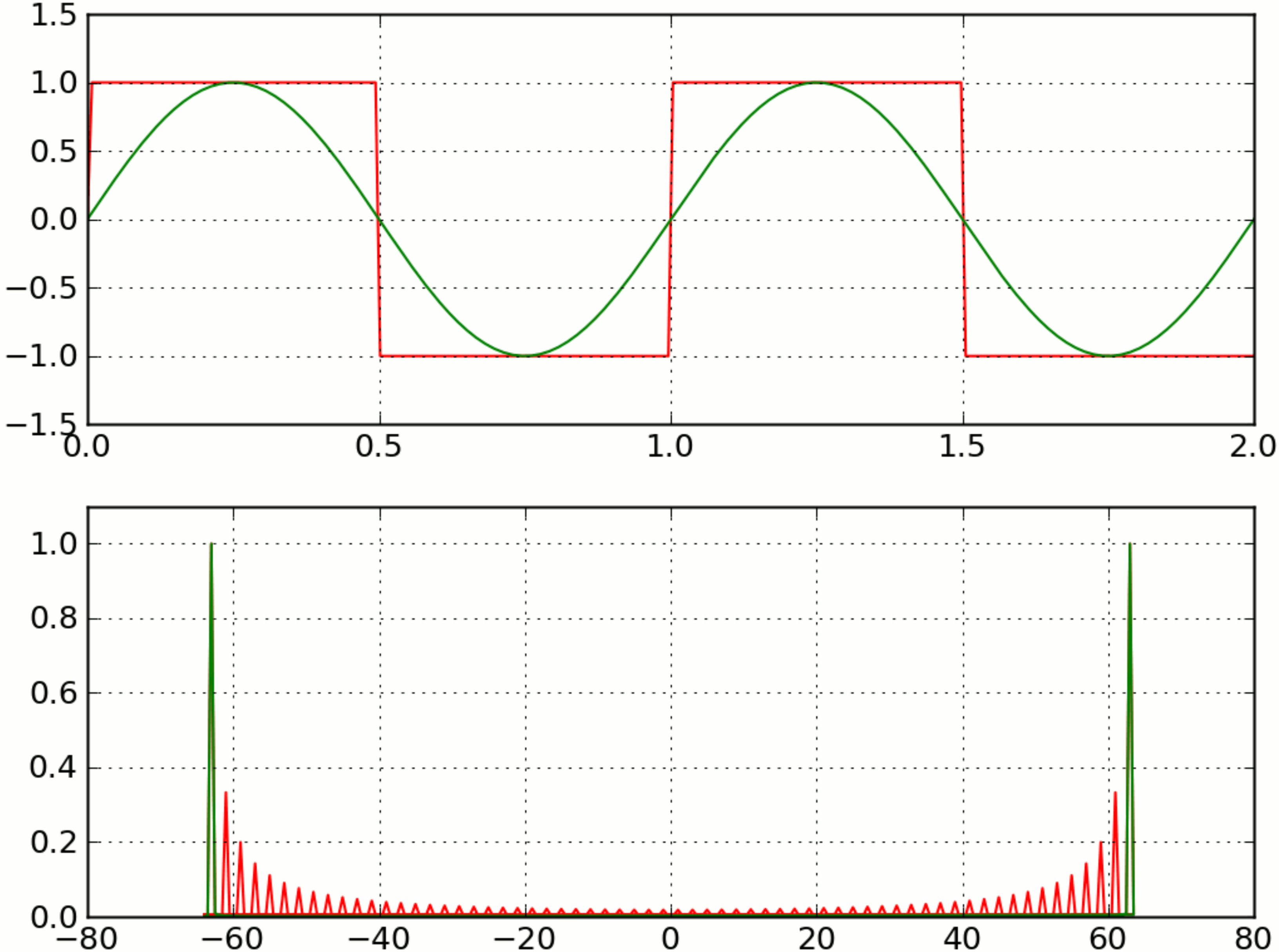
Taylor series → Expand as a (infinite) sum of polynomials

Different Idea: Fourier series → Expand as a (infinite) sum of sinusoids

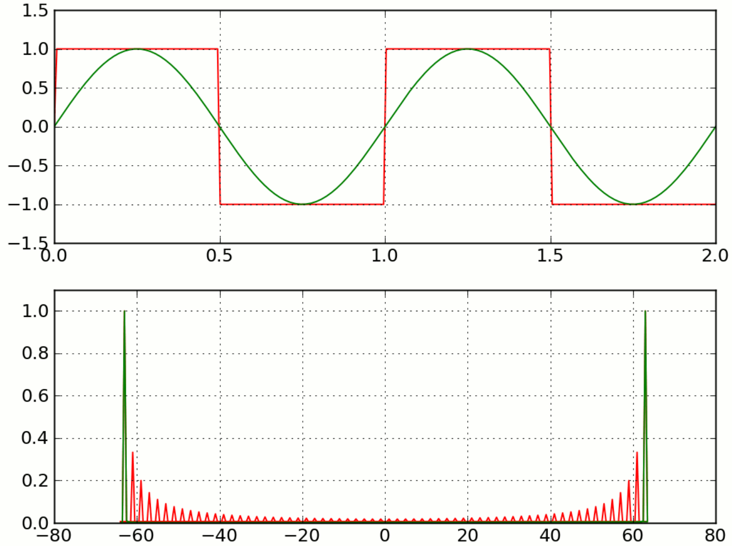
“The exponential function e^x (in blue), and the sum of the first n+1 terms of its Taylor series at 0 (in red).”



wikipedia (Taylor series)



wikipedia (square wave)



"Animation of the additive synthesis of a square wave with an increasing number of harmonics."

"The six arrows represent the first six terms of the Fourier series of a square wave. The two circles at the bottom represent the exact square wave (blue) and its Fourier-series approximation (purple)."

wikipedia (square wave)

Fourier series

$$
\begin{aligned}
f(t) = a_o + a_1 \sin(\omega t) + b_1 \cos(\omega t) + \\
+ a_2 \sin(2\omega t) + b_2 \cos(2\omega t) + \\
+ a_3 \sin(3\omega t) + b_3 \cos(3\omega t) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
= A_0 + A_1 \sin(\omega t + \phi_1) \\
+ A_2 \sin(2\omega t + \phi_2) \\
+ A_3 \sin(3\omega t + \phi_3) + \cdots
\end{aligned}
$$

$$
= \sum_{n=0}^{\infty} A_n \sin(n\omega t + \phi_n)
$$

$$
= \sum_{n=0}^{\infty} B_n e^{in\omega t} \quad \text{where } B_n \in \mathbb{C},\ i = \sqrt{-1}
$$

Complex #s are much more compact and easier to deal with

Fourier series

Dirichlet's theorem: *If $f(t)$ is periodic of period 2π, if for $-\pi < t < \pi$ the function $f(t)$ has a finite number of maximum and minimum values and a finite number of discontinuities, and if $\int_{-\pi}^{\pi} f(t)\,dt$ is finite, then the Fourier series converges to $f(t)$ at all points where $f(t)$ is continuous, and at jump-points it converges to the arithmetic mean of the right-hand and left-hand limits of the function.*

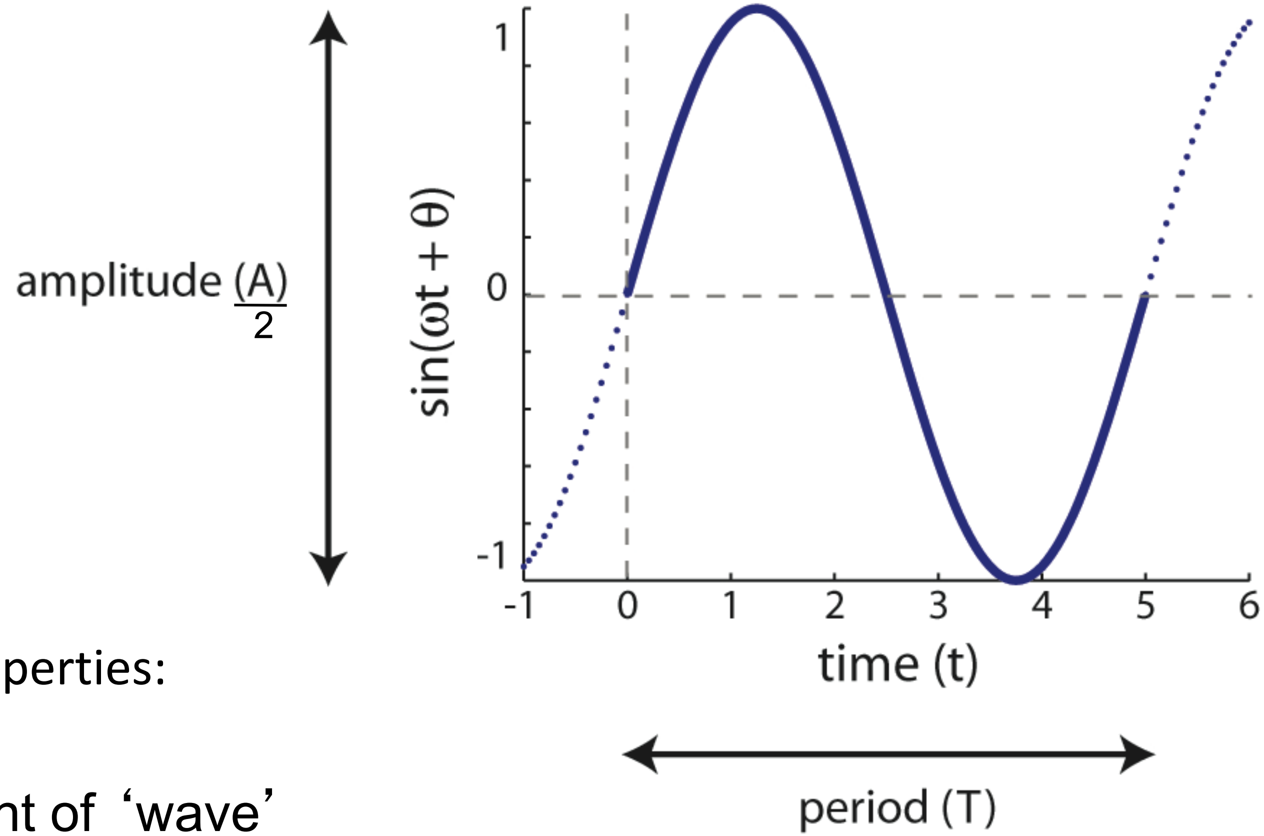
- Mathematically *sufficient* conditions, but not necessarily *necessary*

- Regardless, there is a deceptively simple, yet very powerful idea here:

→ One can describe any sort of function (e.g., a time waveform carrying some information) as a bunch of sinusoids!

Devries (1994)

Review: Trigonometry



A sinusoid has 3 basic properties:

i - Amplitude – height of 'wave'

ii - Frequency – how often does it repeat?

1/T [Hz]

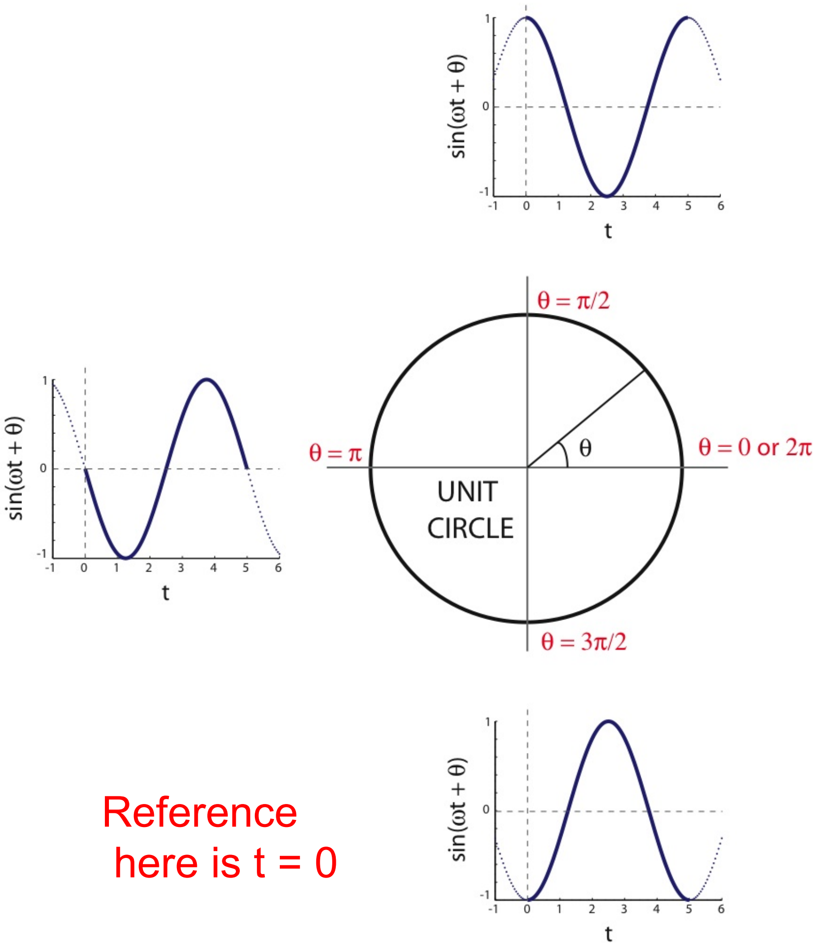
iii - Phase (ϕ) – where is the peak?

needs a reference!

Review: Trigonometry

→ What does phase tell us?

- Phase tells time information
- Analogous to angle around circle
- Arbitrary outside range [0,2π]
  ('phase unwrapping ambiguity')
- 1 cycle = 360^{o} = 2π radians



Reference here is t = 0

Connection to complex numbers??

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad (11.28)$$

Review: Complex numbers

Definition:

$$i = \sqrt{-1} \ \ (= j)$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

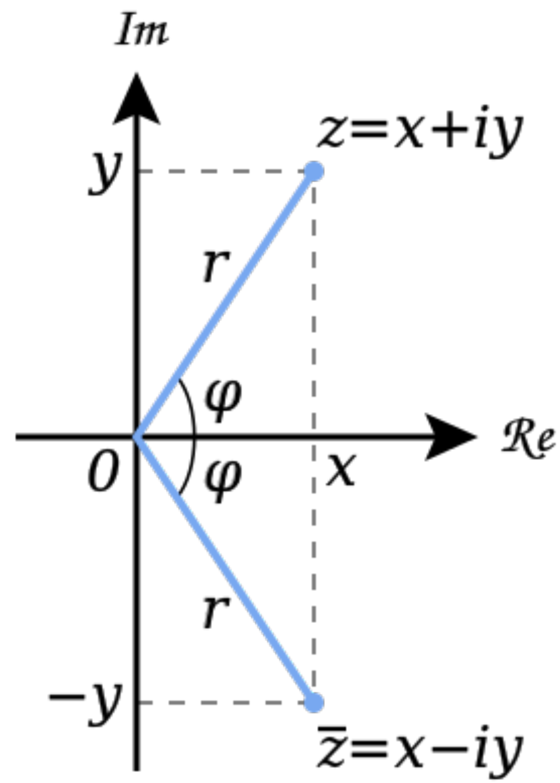
$$i^5 = i$$

$$\ldots$$

Cartesian form:

$$z = a + ib$$

$$\mathrm{Re}(z) = a, \ \ \mathrm{Im}(z) = b$$



Polar form & complex conjugate

Communicative rule:

$$z_1 + z_2 = (a + ib) + (c + id) = (a + b) + i(c + d) = z_3$$

Multiplicative rule:

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc) = z_3$$

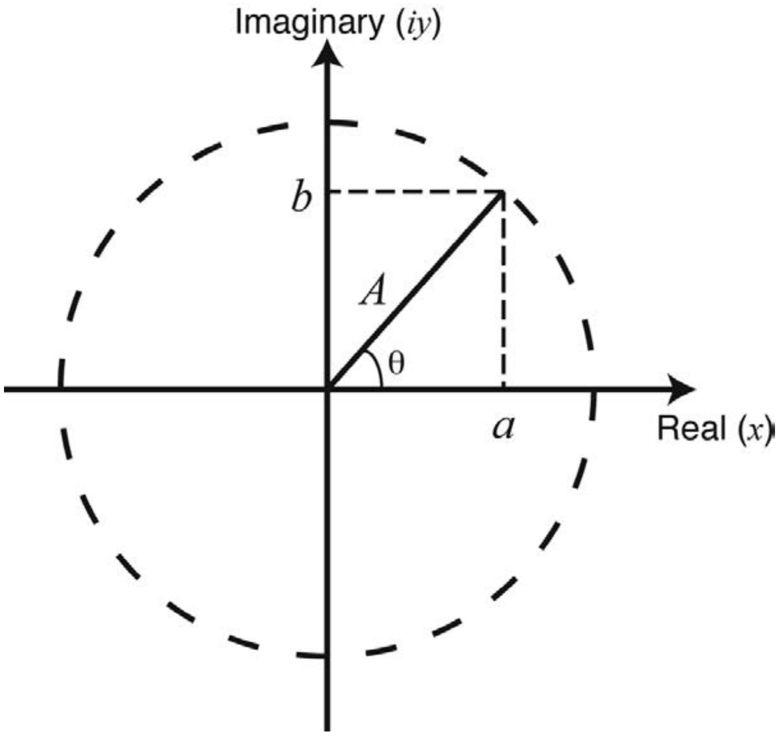
wikipedia (complex number)

Review: Complex numbers

Euler's formula → Polar form

$$a + ib = Ae^{i\theta}$$

$$= A(\cos\theta + i\sin\theta)$$



Cartesian Form

$$a = A\cos(\theta)$$

$$b = A\sin(\theta)$$

⇔

Polar Form

$$A = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

→ Very useful to consider complex numbers geometrically via a circle centered about the origin in the complex plane

Magnitude

$$|a + ib| = A$$

Phase

$$\angle(a + ib) = \theta$$

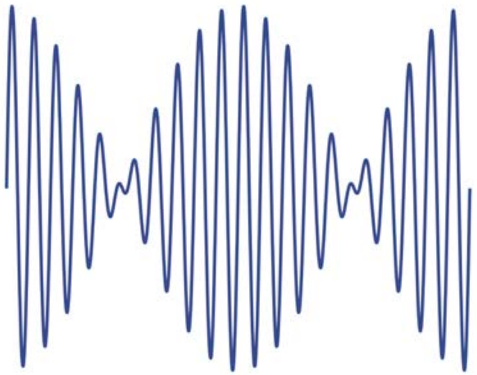
- At the most basic level, simply consider a complex number as a means to compactly express two real numbers (along with the remarkable number i)

Review: Sum of 2 sinusoids

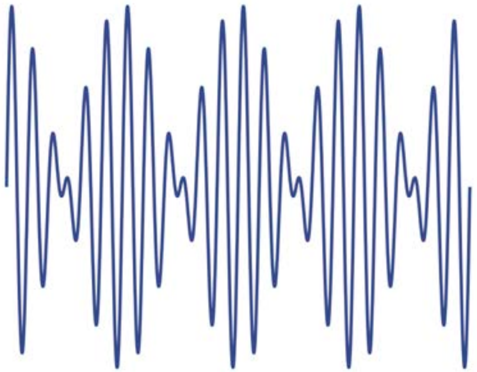
$$x(t) = A_1 \sin(2\pi f_1 t + \phi_1) + A_2 \sin(2\pi f_2 t + \phi_2)$$

→ Different frequency combinations yield different 'patterns'

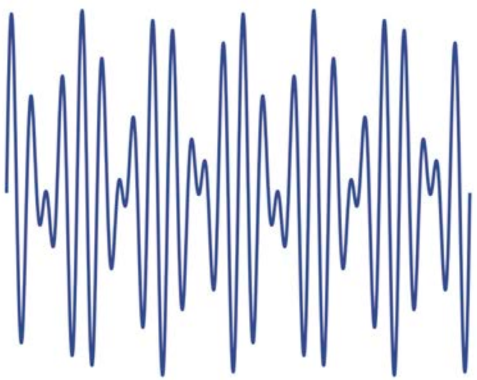
$f_1=1, f_2=1.1$
$A_1=1, A_2=1$
$\phi_1=0, \phi_2=0$



$f_1=1, f_2=1.2$
$A_1=1, A_2=1$
$\phi_1=0, \phi_2=0$



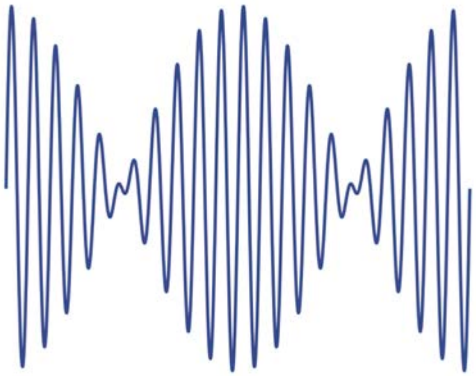
$f_1=1, f_2=1.3$
$A_1=1, A_2=1$
$\phi_1=0, \phi_2=0$



Review: Sum of 2 sinusoids

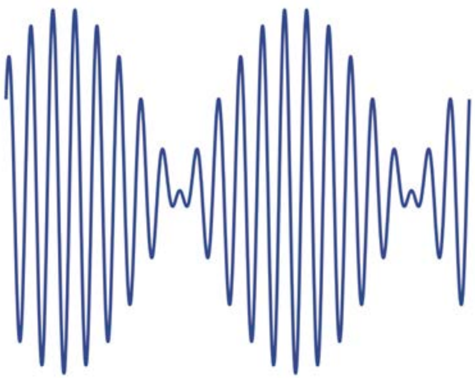
$$x(t) = A_1 \sin(2\pi f_1 t + \phi_1) + A_2 \sin(2\pi f_2 t + \phi_2)$$

$f_1=1,\ f_2=1.1$
$A_1=1,\ A_2=1$
$\phi_1=0,\ \phi_2=0$



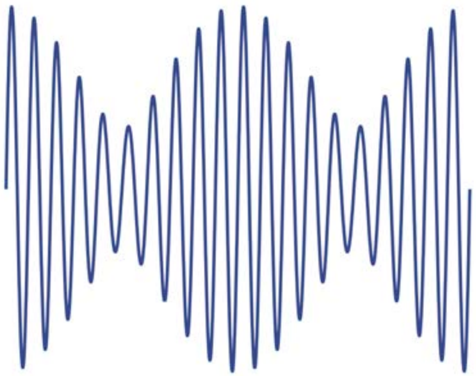
→ Changing (relative) phase affects summation

$f_1=1,\ f_2=1.1$
$A_1=1,\ A_2=1$
$\phi_1=\pi/2,\ \phi_2=0$



→ Changing (relative) amplitudes affects summation

$f_1=1,\ f_2=1.1$
$A_1=2,\ A_2=1$
$\phi_1=0,\ \phi_2=0$



Sinusoids as basis functions

$$\int_{-\pi}^{\pi} \sin mt \sin nt \, dt = \begin{cases} \pi\delta_{m,n}, & m \neq 0, \\ 0, & m = 0, \end{cases}$$

- Sinusoids make a good choice for basis functions, as they are 'complete' in that they are orthogonal to one another over the interval [0,2π]

$$\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \begin{cases} \pi\delta_{m,n}, & m \neq 0, \\ 2\pi, & m = n = 0, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \cos nt \, dt = 0, \quad \text{all integral } m \text{ and } n.$$

→ The sum of the product over the interval is zero for disparate frequencies (i.e., they cancel one another out!)

→ *Similar idea as orthogonal unit vectors in coordinate space*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

Fourier coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt.$$

- This provides a 'recipe' for figuring out the appropriate weighting for each term

Devries (1994)

Slightly different formulation/notation...

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \qquad x \in (-\pi, \pi]$$

- For arguments sake, multiply both sides by $\cos mx$ and integrate over interval [-π,π]

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx \right)$$

- The orthogonality conditions (below) thereby make it easy to determine the associated Fourier coefficients

Orthogonality relationships

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \qquad \forall\ n, m$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m. \end{cases}$$

Fourier coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \qquad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \qquad n > 0$$

Kutz (2012)

Fourier series: Complex notation

$$f(t) = a_o + a_1 \sin(\omega t) + b_1 \cos(\omega t) + \\ + a_2 \sin(2\omega t) + b_2 \cos(2\omega t) + \\ + a_3 \sin(3\omega t) + b_3 \cos(3\omega t) + \cdots$$

For each frequency, write as a sin and a cos...

$$= A_0 + A_1 \sin(\omega t + \phi_1) \\ + A_2 \sin(2\omega t + \phi_2) \\ + A_3 \sin(3\omega t + \phi_3) + \cdots$$

... or as a single sinusoid (along with a phase)....

$$= \sum_{n=0}^{\infty} A_n \sin(n\omega t + \phi_n)$$

$$= \sum_{n=0}^{\infty} B_n e^{in\omega t} \quad \text{where } B_n \in \mathbb{C},\ i = \sqrt{-1}$$

... or as a complex number

Complex #s are much more compact and easier to deal with

Fourier series: Complex notation

Real

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt.$$

Complex

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

$$c_n = \begin{cases} (a_n - ib_n)/2, & n > 0, \\ a_0/2, & n = 0, \\ (a_{|n|} + ib_{|n|})/2, & n < 0. \end{cases}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt.$$

- Two descriptions are equivalent
- One fine point: distinction between continuous and discrete

Devries (1994)

Fourier transforms

- *Fourier series* are useful for describing functions over a limited region or on the infinite interval (-∞, ∞), assuming the function is periodic

- *Fourier transforms* are useful for describing non-periodic functions on the infinite interval (most intervals we deal with in the 'real world' are not infinite, a point we will come back to)

To develop the transform, let's first consider the series representation of a function that is periodic on the interval $[-T, T]$. Making the substitution $t \to \pi t/T$, we have

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/T},$$

where

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-in\pi t/T} \, dt$$

The discrete frequencies in sum are: $\omega = \frac{n\pi}{T}$

With the successive differences being: $\Delta\omega = \frac{\pi}{T}$

Devries (1994)

Fourier transforms

(complex) Fourier coefficients

We then rewrite our series as:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\Delta\omega t} \qquad c_n = \frac{\Delta\omega}{2\pi} \int_{-T}^{T} f(t) e^{-in\Delta\omega t}\, dt$$

Now we define:

$$c_n = \frac{\Delta\omega}{\sqrt{2\pi}} g(n\Delta\omega)$$

$$g(n\Delta\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} f(t) e^{-in\Delta\omega t}\, dt$$

Which leads us to:

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \Delta\omega\, g(n\Delta\omega)\, e^{in\Delta\omega t}$$

Taking the limit where $T \rightarrow \infty$:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

Note: $n\Delta\omega$ becomes (continuous variable) ω

Devries (1994)

Fourier transforms

We now *define* $g(\omega)$ to be the Fourier transform of $f(t)$,

$$\mathcal{F}[f(t)] = g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt,$$

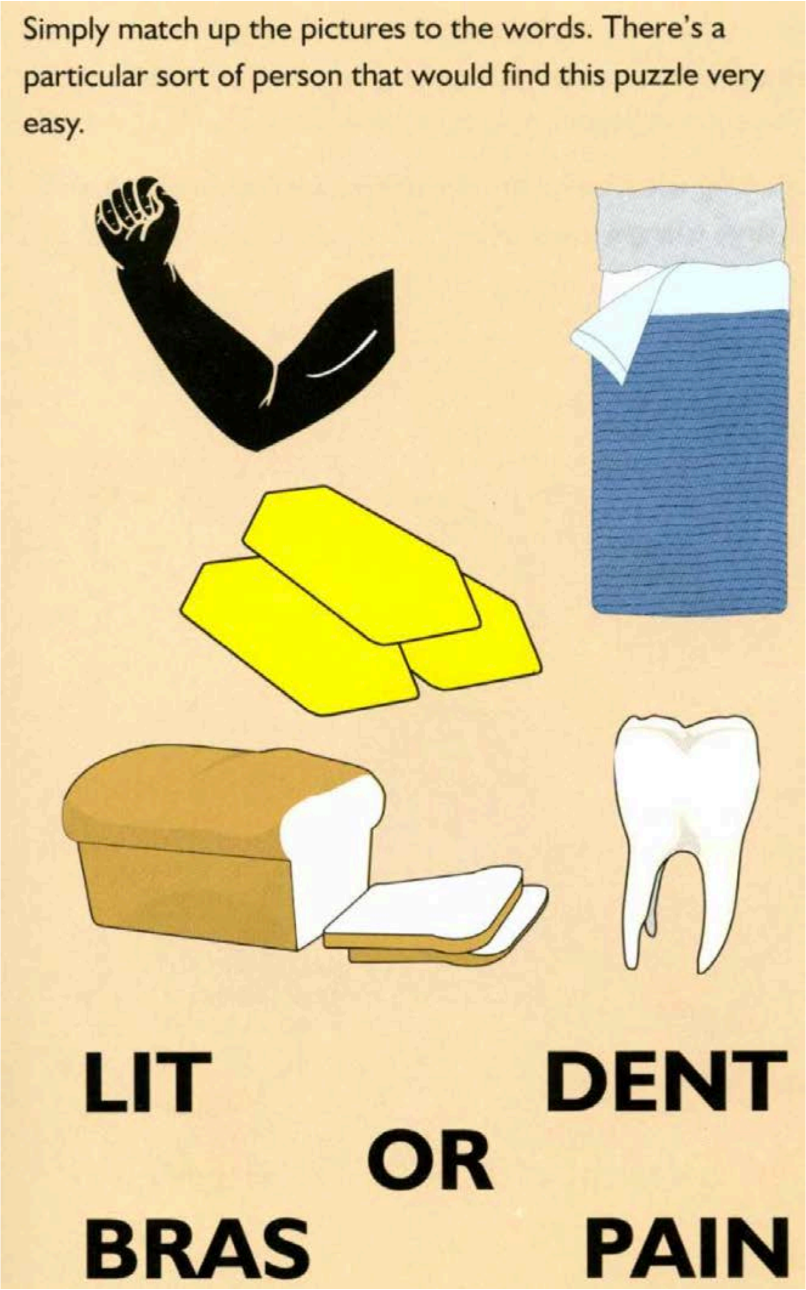
and $f(t)$ to be the *inverse* transform of $g(\omega)$,

$$\mathcal{F}^{-1}[g(\omega)] = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega.$$

- Seemingly abstract/simple idea has vast implications in terms of how we encode and decipher information (e.g., signal processing) as well as mathematical methods in physics and linear systems theory

Devries (1994)

Summary



- Think of 'transforming' in the context of encoding '*information*' (e.g., different languages)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$\mathcal{F}[f(t)] = g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$