Biophysics I (BPHS 3090)

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Cell membrane acts like an RC filter

Figure 1.8
Axon behaves in fashion similar to a leaky submarine cable.
Cable Model - Overview

→ Combine together both “models”
For $\Delta V_m$ small:

\[
K_m = 2\pi a J_m = 2\pi a C_m \frac{dV_m}{dt} + 2\pi a G_m (V_m - V_m^o) = c_m \frac{dV_m}{dt} + g_m (V_m - V_m^o)
\]

Combine with core-conductor model:

\[
\frac{\partial^2 V_m}{\partial z^2} = (r_o + r_i) K_m - r_o K_e = (r_o + r_i) \left[ c_m \frac{\partial V_m}{\partial t} + g_m (V_m - V_m^o) \right] - r_o K_e
\]

\[
V_m + \frac{c_m}{g_m} \frac{\partial V_m}{\partial t} - \frac{1}{g_m (r_o + r_i)} \frac{\partial^2 V_m}{\partial z^2} = V_m^o + \frac{r_o}{g_m (r_o + r_i)} K_e
\]
Introduce two new constants ($\tau_M$ and $\lambda_C$)

$$V_m + \frac{c_m}{g_m} \frac{\partial V_m}{\partial t} - \frac{1}{g_m(r_o + r_i)} \frac{\partial^2 V_m}{\partial z^2} = V_m^o + \frac{r_o}{g_m(r_o + r_i)} K_e$$

Let $V_m = v_m + V_m^o$:

$$(\text{incremental change in memb. potential})$$

$$v_m + \tau_M \frac{\partial v_m}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m}{\partial z^2} = r_o \lambda_C^2 K_e \quad (\text{Cable Equation})$$
Cable Equation

Let \( v_m(z, t) = V_m(z, t) - V_m^o \) and \( |v_m(z, t)| << |V_m^o| \):

\[
v_m(z, t) + \tau_M \frac{\partial v_m(z, t)}{\partial t} - \chi_C^2 \frac{\partial^2 v_m(z, t)}{\partial z^2} = r_o \chi_C^2 K_e(z, t)
\]

Note:
Somewhat similar to the diffusion equation (but not exactly due to extra \( v_m \) term)
Constants: $\tau_M$ and $\lambda_C$

$$V_m + \tau_M \frac{\partial V_m}{\partial t} - \lambda_C^2 \frac{\partial^2 V_m}{\partial z^2} = V_m^o + r_o \lambda_C^2 K_e$$

**Space constant** ($\lambda_C$) - property of cell, not just membrane

$$\lambda_C = \frac{1}{\sqrt{(r_i + r_o) g_m}} \approx \sqrt{\frac{a}{2\rho_i G_m}}$$ (assuming $r_o << r_i$)

Wider axons $\Rightarrow$ Further propagation/less degradation

**Time constant** ($\tau_M$) – independent of cellular dimensions

$$\tau_M = \frac{c_m}{g_m}$$
Cable Equation

Let $v_m(z, t) = V_m(z, t) - V^o_m$ and $|v_m(z, t)| << |V^o_m| :$

$$v_m(z, t) + \tau_M \frac{\partial v_m(z, t)}{\partial t} - \lambda_C^2 \frac{\partial^2 v_m(z, t)}{\partial z^2} = r_o \lambda_C^2 K_e(z, t)$$

Figure 3.8

Axon $\leftrightarrow$ Leaky submarine ‘cable’
Cable Model – Solution for spatial impulse

"match boundary conditions" at $z=0$

$A$

$z$

$\frac{dv_m(z)}{dz}$

$\frac{A}{\lambda_C}$

$-\frac{A}{\lambda_C}$

$\frac{d^2v_m(z)}{dz^2}$

$\frac{A}{\lambda_C^2}$

$\left(-\frac{2A}{\lambda_C}\right)$

$z$

Figure 3.10

Figure 3.9

$0 \quad z$

Figure 3.11

Amplitude falls off (re space const.)
→ Space constant ($\lambda_c$) typically on order of mm (even less for small unmyelinated fibers)

→ Solutions allow for propagation, but in a decremental fashion

→ Axons alone are not good ‘cables’ for sending signals long-ish distances!
Cable Model – Solution for temporal & spatial impulse

Assume infinitesimal electrode and \( i_e(t) \) brief so that

\[
k_e(z, t) = 0; \quad \text{if} \ z \neq 0 \text{ or } t \neq 0.
\]

For \( t \neq 0 \) or \( z \neq 0 \)

\[
v_m(z, t) + \tau_M \frac{\partial v_m}{\partial t} - \lambda_C \frac{\partial^2 v_m}{\partial z^2} = 0
\]

Let

\[
v_m(z, t) = w(z, t) e^{-t/\tau_M}
\]

Then

\[
\frac{\partial v_m}{\partial t} = -\frac{1}{\tau_M} w(z, t) e^{-t/\tau_M} + \frac{\partial w}{\partial t} e^{-t/\tau_M}
\]

\[
\frac{\partial^2 v_m}{\partial z^2} = \frac{\partial^2 w}{\partial z^2} e^{-t/\tau_M}
\]
Cable Model – (A) Solution

Substituting,

\[ w(z, t)e^{-t/\tau_M} = w(z, t)e^{-t/\tau_M} + \tau_M \frac{\partial w}{\partial t} e^{-t/\tau_M} - \lambda_C^2 \frac{\partial^2 w}{\partial z^2} e^{-t/\tau_M} = 0 \]

\[ \tau_M \frac{\partial w}{\partial t} = \lambda_C^2 \frac{\partial^2 w}{\partial z^2} \]

Solving cable equation (here w/ change of variable) is like diffusion equation!

![Figure 3.23](image1)

![Figure 3.24](image2)
Cable Model – (A) Solution

Figure 3.25
Solutions allow for propagation, but in a decremental fashion.
Cellular dimensions re space constant ($\lambda_c$) determine whether a cell is electrically *small* or *large*.
Ex.

Electrode 3 likely closest to end-plate

Figure 3.33
Linearity \rightarrow Superposition
Key considerations with regard to synapses (i.e., inter-neuron communication)

“Electronic distance”

“Temporal integration”
Spatial integration

Figure 3.36
Looking Ahead: Hodgkin-Huxley

Decremental conduction

Decrement-free conduction

Electrically inexcitable cell

Electrically excitable cell
Hodgkin Huxley model

Figure 4.7
\[ G_K(V_m, t) = \overline{G}_K n^4(V_m, t) \]
\[ G_{Na}(V_m, t) = \overline{G}_{Na} m^3(V_m, t) h(V_m, t) \]
\[ n(V_m, t) + \tau_n(V_m) \frac{dn(V_m, t)}{dt} = n_\infty(V_m) \]
\[ m(V_m, t) + \tau_m(V_m) \frac{dm(V_m, t)}{dt} = m_\infty(V_m) \]
\[ h(V_m, t) + \tau_h(V_m) \frac{dh(V_m, t)}{dt} = h_\infty(V_m) \]