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Summary: HH Equations

\[
\frac{1}{2\pi a(r_o + r_i)} \frac{\partial^2 V_m}{\partial z^2} = C_m \frac{\partial V_m}{\partial t} + G_K(V_m, t) (V_m - V_K) + G_{Na}(V_m, t) (V_m - V_{Na}) + G_L(V_m - V_L)
\]

\[
G_K(V_m, t) = \overline{G}_K n^4(V_m, t)
\]

\[
G_{Na}(V_m, t) = \overline{G}_{Na} m^3(V_m, t) h(V_m, t)
\]

\[
n(V_m, t) + \tau_n(V_m) \frac{dn(V_m, t)}{dt} = n_\infty(V_m)
\]

\[
m(V_m, t) + \tau_m(V_m) \frac{dm(V_m, t)}{dt} = m_\infty(V_m)
\]

\[
h(V_m, t) + \tau_h(V_m) \frac{dh(V_m, t)}{dt} = h_\infty(V_m)
\]

\[
\tau_x \frac{dx}{dt} + x = x_\infty \quad \frac{dx}{dt} = \alpha_x (1 - x) - \beta_x x
\]

\[
x_\infty = \alpha_x / (\alpha_x + \beta_x) \text{ and } \tau_x = 1 / (\alpha_x + \beta_x)
\]

\[
\alpha_m = \frac{-0.1(V_m + 35)}{e^{-0.1(V_m+35)} - 1},
\]

\[
\beta_m = 4e^{-(V_m+60)/18},
\]

\[
\alpha_h = 0.07e^{-0.05(V_m+60)},
\]

\[
\beta_h = \frac{1}{1 + e^{-0.1(V_m+30)}},
\]

\[
\alpha_n = \frac{-0.01(V_m + 50)}{e^{-0.1(V_m+50)} - 1},
\]

\[
\beta_n = 0.125e^{-0.0125(V_m+60)}
\]
(numerically) Solving HH Eqns.

SoftCell numerically integrates the ODEs (e.g., Euler method, Runge-Kutta)

How to run HH model backwards?

\[ J_m(t) \uparrow \quad \cdots \quad J_o \]

\[ \delta \ll \tau_m, \tau_n, \tau_h \]

\[ V_m(t) \uparrow \quad \cdots \quad V^o_m \]

\[ \Delta V \approx J_o \delta / C_m \]

\[ m(t) \uparrow \]

\[ \frac{dm}{dt} = \frac{m^\infty - m}{\tau_m} \]

\[ m^\infty(V_m) \]

\[ n(t) \uparrow \]

\[ \frac{dn}{dt} = \frac{n^\infty - n}{\tau_n} \]

\[ n^\infty(V_m) \]

\[ h(t) \uparrow \]

\[ \frac{dh}{dt} = \frac{h^\infty - h}{\tau_h} \]

\[ h^\infty(V_m) \]
Finally there was the difficulty of computing the action potentials from the equations which we had developed. We had settled all the equations and constants by March 1951 and hoped to get these solved on the Cambridge University computer. However, before anything could be done we learnt that the computer would be off the air for 6 months or so while it underwent a major modification. Andrew Huxley got us out of that difficulty by solving the differential equations numerically using a hand-operated Brunsviga. The propagated action potential took about three weeks to complete and must have been an enormous labour for Andrew. But it was exciting to see it come out with the right shape and velocity and we began to feel that we had not wasted the many months that we had spent in analysing records.

—Hodgkin, 1977
Propagated APs

$V_m(z, t) - V_m^0$ (mV)

$t$ (ms)

Figure 4.30

→ Solutions only stable for appropriate choice of conduction velocity
(think back to cable model; $C_m$ matters!)

Figure 4.31
Propagated APs

Model Solutions

Stimulus
(think cable model)

Figure 4.29
Four phases:

1. Local disturbance due to capacitance
   (behaves like cable model)

2. Onset: $V_m$ change triggers $m$
   (increased $G_{Na}$ take $V_m$ with it)

3. Falloff: $h$ turns off, $n$ turns on
   (both work to lower $V_m$ back towards $V_k$, basis for absolute refractory period)

4. Undershoot: increased $G_k$ pushes $V_m$ beyond $V_{om}$
   (basis for relative refractory period)

Note: Membrane current ($J_M$) can be parsed up into two components: a capacitive current ($J_C$) and an ionic current ($J_{ion}$)
Note: Fairly little net current across membrane
(i.e., relatively few net ions transported)
Note lag between $V_m$ and $G_m$ (stems from capacitive surge)

Similar picture for propagated AP

Figure 4.32
Note lag between $V_m$ and $G_m$ (stems from capacitive surge)
Threshold

**In vivo:** For the same stimulus, sometimes an AP fires, sometimes it does not.

What is mechanism for a threshold?

→ Model exhibits ‘exceedingly narrow threshold region’

Note: Model is deterministic and does not capture stochastic behaviors manifest in vivo.
Threshold

Note lag for AP to occur (stems from capacitive build-up to threshold)

$V_m(t) - V^0_m$ (mV)

Time (ms)

Figure 4.42
Determine $J_{ion} - V_m$ relationship right after shock (dashed line)

- Current purely due to $C_m$
- Membrane “deciding” whether to fire AP or not

\[ J_{ion} = -J_C = -C_m \frac{dV_m}{dt} \]

Figure 4.42

Note: This picture only holds as a snapshot right after the stimulus

Figure 4.43
- Equilibrium points
- Stability
- Threshold
- Ohm’s Law: Negative resistance?
These pictures make it easy to envision a stochastic component too (e.g., consider random force jittering object about)
Threshold

assume \( n \) and \( h \) are constant

→ Ultimately more than one ion is needed
  (\( \text{Na}^+ \) alone is insufficient)
Threshold: Phase Plane Portrait

assumes $n$ and $h$ are constant, but $m$ varies dynamically
Refractory Period

**Figure 1.13**

**Figure 4.52**

**Figure 4.53**

\[ V(t) = V_m(t) - V_m^o \ (mV) \]