Biophysics I (BPHS 4080)

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Model for electrically large cells → Core-Conductor Model (starting point)

*Note:* Applicable regardless of whether or not cell is electrically excitable
Core Conductor Model

Current through inner conductor

\[ R_i = r_i \, dz \]

Current through outer conductor

\[ R_o = r_o \, dz \]

Current through membrane

\[ I_m = k_m \, dz \]
Core Conductor Model

Figure 2.7
Figure 2.6  Geometry of the core conductor model of a cylindrical cell.
Figure 2.6  Geometry of the core conductor model of a cylindrical cell.
Assumption & Variables in the Core Conductor Model

- The cell membrane is a cylindrical boundary that separates two conductors of electric current, the intracellular and extracellular solutions, which are assumed to be homogeneous and isotropic and to obey Ohm’s law.
- All the electrical variables have cylindrical symmetry, i.e., all the electrical variables are independent of $\theta$.
- A circuit theory description of currents and voltages is adequate. That is, the quasi-static terms of Maxwell’s equations are sufficient, and electromagnetic radiation effects are negligible.
- Currents in the inner and outer conductors flow in the longitudinal direction only. Current flows through the membrane in the radial direction only.
- At a given longitudinal position along the cell, the inner and outer conductors are equipotentials, so the only variation in potential in the radial direction, $r$, occurs across the membrane.

The variables used to describe the electrical properties of a cylindrical cell are defined as follows:

- $I_o(z, t)$ is the total longitudinal current flowing in the positive $z$-direction in the outer conductor (A).
- $I_i(z, t)$ is the total longitudinal current flowing in the positive $z$-direction in the inner conductor (A).
- $J_m(z, t)$ is the membrane current density flowing from the inner conductor to the outer conductor (A/m$^2$).
- $K_m(z, t)$ is the membrane current per unit length flowing from the inner conductor to the outer conductor (A/m).
- $K_e(z, t)$ is the current per unit length due to external sources applied in a cylindrically symmetric manner (A/m). Inclusion of this current allows us to represent the current applied through external electrodes to the cell surface. A similar term could be added to represent the current supplied by an internal electrode (see Problem 2.6).
- $V_m(z, t)$ is the membrane potential, which is a positive quantity when the inner conductor has a positive potential with respect to the outer conductor (V).
- $V_i(z, t)$ is the potential in the inner conductor (V).
- $V_o(z, t)$ is the potential in the outer conductor (V).
- $r_o$ is the resistance per unit length of the outer conductor ($\Omega$/m).
- $r_i$ is the resistance per unit length of the inner conductor ($\Omega$/m).
- $a$ is the radius of the cylindrical cell.
KCL at (a): \[ I_i(z, t) - I_i(z + \Delta z, t) - K_m(z, t) \Delta z = 0 \]

KCL at (d): \[ I_o(z, t) - I_o(z + \Delta z, t) + K_m(z, t) \Delta z - K_e(z, t) \Delta z = 0 \]

Ohm's law at (a) − (b): \[ V_i(z, t) - V_i(z + \Delta z, t) = r_i \Delta z I_i(z + \Delta z, t) \]

Ohm's law at (c) − (d): \[ V_o(z, t) - V_o(z + \Delta z, t) = r_o \Delta z I_o(z + \Delta z, t) \]
\[
\frac{I_i(z + \Delta z, t) - I_i(z, t)}{\Delta z} = -K_m(z, t) \rightarrow \frac{\partial I_i(z, t)}{\partial z} \\
\frac{I_o(z + \Delta z, t) - I_o(z, t)}{\Delta z} = K_m(z, t) - K_e(z, t) \rightarrow \frac{\partial I_o(z, t)}{\partial z} \\
\frac{V_i(z + \Delta z, t) - V_i(z, t)}{\Delta z} = -r_i I_i(z + \Delta z, t) \rightarrow \frac{\partial V_i(z, t)}{\partial z} \\
\frac{V_o(z + \Delta z, t) - V_o(z, t)}{\Delta z} = -r_o I_o(z + \Delta z, t) \rightarrow \frac{\partial V_o(z, t)}{\partial z}
\]
Core – Conductor Equations

\[ \frac{\partial I_i(z, t)}{\partial z} = -K_m(z, t) \]

\[ \frac{\partial I_o(z, t)}{\partial z} = K_m(z, t) - K_e(z, t) \]

\[ \frac{\partial V_i(z, t)}{\partial z} = -r_i I_i(z, t) \]

\[ \frac{\partial V_o(z, t)}{\partial z} = -r_o I_o(z, t) \]

\[ V_m(z, t) = V_i(z, t) - V_o(z, t) \]

\[ \frac{\partial V_m(z, t)}{\partial z} = \frac{\partial V_i(z, t)}{\partial z} - \frac{\partial V_o(z, t)}{\partial z} = -r_i I_i(z, t) + r_o I_o(z, t) \]

\[ \frac{\partial^2 V_m(z, t)}{\partial z^2} = -r_i \frac{\partial I_i(z, t)}{\partial z} + r_o \frac{\partial I_o(z, t)}{\partial z} = r_i K_m(z, t) + r_o (K_m(z, t) - K_e(z, t)) \]
THE Core – Conductor Equation

\[
\frac{\partial^2 V_m(z, t)}{\partial z^2} = (r_o + r_i) K_m(z, t) - r_o K_e(z, t)
\]
Assumptions/geometry above, along with Kirchoff’s & Ohm’s Laws lead us to the...

THE Core – Conductor Equation

\[
\frac{\partial^2 V_m(z, t)}{\partial z^2} = (r_o + r_i)K_m(z, t) - r_oK_e(z, t)
\]

→ Relates spatial change in transmembrane potential to current flowing through the membrane
Some Implications

Consider no external electrodes (i.e., $K_e = 0$):

$$K_m(z, t) = \frac{1}{r_o + r_i} \frac{\partial^2 V_m(z, t)}{\partial z^2}$$

Conservation of charge requires:

$$I_i(z, t) + I_o(z, t) = 0$$

Core – Conductor Equations

$$\frac{\partial I_i(z, t)}{\partial z} = -K_m(z, t)$$

$$\frac{\partial I_o(z, t)}{\partial z} = K_m(z, t) - K_e(z, t)$$

$$\frac{\partial V_i(z, t)}{\partial z} = -r_i I_i(z, t)$$

$$\frac{\partial V_o(z, t)}{\partial z} = -r_o I_o(z, t)$$

$$I_o(z, t) = \frac{1}{r_o + r_i} \frac{\partial V_m(z, t)}{\partial z}$$