**Biophysics I** (BPHS 4080)

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**Website:** http://www.yorku.ca/cberge/4080W2018.html
Assumptions/geometry above, along with Kirchoff’s & Ohm’s Laws lead us to the...

\[ \frac{\partial^2 V_m(z, t)}{\partial z^2} = (r_o + r_i)K_m(z, t) - r_oK_e(z, t) \]

→ Relates spatial change in transmembrane potential to current flowing through the membrane
Some Implications

Consider no external electrodes (i.e., $K_e = 0$):

$$K_m(z, t) = \frac{1}{r_o + r_i} \frac{\partial^2 V_m(z, t)}{\partial z^2}$$

Conservation of charge requires:

$$I_i(z, t) + I_o(z, t) = 0$$

Core – Conductor Equations

$$\frac{\partial I_i(z, t)}{\partial z} = -K_m(z, t)$$
$$\frac{\partial I_o(z, t)}{\partial z} = K_m(z, t) - K_e(z, t)$$
$$\frac{\partial V_i(z, t)}{\partial z} = -r_i I_i(z, t)$$
$$\frac{\partial V_o(z, t)}{\partial z} = -r_o I_o(z, t)$$

$$I_o(z, t) = \frac{1}{r_o + r_i} \frac{\partial V_m(z, t)}{\partial z}$$
Some useful interrelationships...

Relation between extracellular and intracellular potentials

KVL:

\[ v_{m}(z_2) + \Delta v_0 - v_{m}(z_1) - \Delta v_i = 0 \]

\[ \Delta v_o - \Delta v_i = v_{m}(z_1) - v_{m}(z_2) \]

\[ \Delta v_i = - \int_{z_1}^{z_2} r_i I_i dz = -r_i \int_{z_1}^{z_2} I_i dz \]

\[ \Delta v_o = - \int_{z_1}^{z_2} r_o I_o dz = -r_o \int_{z_1}^{z_2} I_o dz \]

\[ I_o = -I_i \]

\[ \Delta v_i = -r_i \int_{z_1}^{z_2} I_i dz = -r_i \int_{z_1}^{z_2} (-I_o) dz = -\frac{r_i}{r_o} \Delta v_o \]

\[ \Delta v_o - \Delta v_i = \Delta v_o + \frac{r_i}{r_o} \Delta v_o = v_{m}(z_1) - v_{m}(z_2) \]

\[ \Delta v_o = \frac{r_o}{r_o + r_i} (v_{m}(z_1) - v_{m}(z_2)) \]

let \( z_1 \to -\infty \): then \( v_o(z_1) \to 0 \) and \( v_m(z_1) \to V_m^o \)

\[ \Delta v_o = v_o(z_2) - v_o(z_1) = v_o(z_2) = \frac{r_o}{r_o + r_i} (V_m^o - v_{m}(z_2)) \]

\[ v_o(z_2) = -\frac{r_o}{r_o + r_i} (v_{m}(z_2) - V_m^o) \]
Propagation at *Constant Velocity*

**Assumption:** Membrane potential behaves in a wave-like fashion (const. velocity)

\[ V_m(z, t) = f \left( t \pm \frac{z}{v} \right) \]

- **Axon**
- Propagation direction of action potential (speed \( v \))

**Note:** We make a bio-inspired assumption as to the form (i.e., shape) of \( f \) here.
Propagation at \textit{Constant Velocity}

\[ V_m(z, t) = f \left( t \pm \frac{z}{v} \right) \]

\begin{itemize}
  \item \textit{“Snapshot” at one specific time} \quad \rightarrow \textit{Spatial dependence}
  \item \textit{“Snapshots” at two different spatial locations} \quad \rightarrow \textit{Temporal dependence}
\end{itemize}

Figure 2.13
Propagation at \textit{Constant Velocity}

\[ V_m(z, t) = f \left( t \pm \frac{z}{\nu} \right) \]

\[ \frac{\partial V_m(z, t)}{\partial z} = \pm \frac{1}{\nu} f \left( t \pm \frac{z}{\nu} \right) \quad \text{and} \quad \frac{\partial V_m(z, t)}{\partial t} = \dot{f} \left( t \pm \frac{z}{\nu} \right) \]

\[ \frac{\partial^2 V_m(z, t)}{\partial z^2} = \frac{1}{\nu^2} \frac{\partial^2 V_m}{\partial t^2} \]

\[ I_o(z, t) = \pm \frac{1}{(r_o + r_i) \nu} \frac{\partial V_m(z, t)}{\partial t} \]

\text{and}

\[ K_m(z, t) = \frac{1}{(r_o + r_i) \nu^2} \frac{\partial^2 V_m(z, t)}{\partial t^2} \]

Think carefully about what the diacritical dot means here!

Wave equation
(differential form)

→ So when we assume a wave propagating at constant velocity, the core conductor model yields explicit time relationships as well

\[ I_o(z, t) = \frac{1}{r_o + r_i} \frac{\partial V_m(z, t)}{\partial z} \]
Connecting action potential-type waves to the core conductor model

\[ I_o(z, t) = \pm \frac{1}{(r_o + r_i)\nu} \frac{\partial V_m(z, t)}{\partial t} \]

and

\[ K_m(z, t) = \frac{1}{(r_o + r_i)\nu^2} \frac{\partial^2 V_m(z, t)}{\partial t^2} \]

\[ I_i(z, t) + I_o(z, t) = 0 \]

Figure 2.12
Conduction Velocity (unmyelinated axon)

\[ K_m(z, t) = \frac{1}{(r_o + r_i) \nu^2} \frac{\partial^2 V_m(z, t)}{\partial t^2} \quad J_m(z, t) = K_m(z, t)/(2\pi a) \]

Left-side: constant, only depends upon electrical properties of membrane per unit area

Right-side: constant, velocity depends only upon axon diameter and fluid resistances
Changing resistivity affects conduction velocity.

Conduction Velocity (unmyelinated axon)

\[
\frac{\partial^2 V_m(z, t)}{\partial t^2} = \frac{1}{J_m(z, t)} = 2\pi a (r_o + r_i) v^2
\]

Axon in:

- seawater
- oil
- seawater
- oil

Figure 2.14

Change \( r_o \)
Conduction Velocity (unmyelinated axon)

Change $r_i$

Inner wire essentially acts like a short

\[
\frac{\partial^2 V_m(z, t)}{\partial t^2} \left/ \frac{J_m(z, t)}{J_m(z, t)} \right. = 2\pi a (r_o + r_i) v^2
\]

$\rightarrow$ ‘Space clamp’
Conduction Velocity (unmyelinated axon)

Assume $r_i \gg r_o$

$$2\pi ar_i v^2 = \kappa_m$$

$$r_i = \frac{\rho_i}{\pi a^2}$$

$$v = \sqrt{\frac{\kappa_m a}{2\rho_i}}$$

→ thicker axons = faster propagation
Core Conductor Model

Figure 2.7
Core-Conductor Model (starting point) → Model for electrically large cells
No assumptions made about membrane!

\[ \frac{\partial^2 V_m(z,t)}{\partial z^2} = (r_o + r_i)K_m(z,t) - r_oK_e(z,t) \]