**BPHS 4090 (Fall 2013) - Computer Exercise 7: Nonlinearity and Fractals**

**Due Date: Dec. 4, 2013 5:00 PM**

# Background

Many aspects of biological systems are nonlinear. As such, interesting and unexpected challenges arise that may not be expected when dealing with linear systems (e.g., chaos). Powerful insights can be gained by studying relatively simple nonlinear systems and their behavior. One example of this is pattern formation (i.e., fractals). The goal here is to examine the *quadratic map*. A map is a simplified version of a dynamical system (e.g., a differential equation) where there is a discrete iteration. For example, consider a simple map given by

$$x_{n+1} = x_n + 1 \tag{1}$$

where $n \in \mathbb{Z}$ (i.e., positive integers, including 0). Start with some initial value $x_0$, then 'iterate' the map. For example, starting with $x_0 = 2$, then $x_1 = 3$, $x_2 = 4$, $x_3 = 5$, and so on (here $n$ represents the $n$'th iteration of the map). The quadratic map is given as

$$x_{n+1} = a - x_n^2 \tag{2}$$

where $a \in \mathbb{C}$ (i.e., an arbitrary complex number). Another nonlinear map of interest is the logistic map, given by

$$x_{n+1} = r x_n (1 - x_n) \tag{3}$$

here $r$ is a positive (real) number.

An important concept is that of a fixed point. These are points which do not change under subsequent iterations of the map. We write this as $x_n + k = x_n$, where $k$ is an integer and we define the fixed point to be of period $k$. These are useful when dealing with nonlinear systems, as we linearize about fixed points and determine the behavior in the neighborhood of that point (e.g., via the Jacobian). Fixed points are generally classified into four categories: stable, unstable, saddle, or a center.

The behavior of the logistic map depends strongly upon the choice of $r$. As $r$ varies, the qualitative behavior of the map changes. This gives to what is known as the *period doubling cascade*, an indicator of a transition into chaos.

# Questions

a. Create a code to reproduce the period doubling cascade for the logistic map. This can be done by considering a range over $r$ (say 2 to 4, varied in a hundred steps between), choosing an initial condition (say $x_0 = 1$), then for each value of $r$ iterating the map for a sufficient number of cycles to allow for settling of transients (e.g., settling into a fixed point; say $n_S = 50$), then iterate for $N$ more cycles after that to see how the dynamics vary (say $N$=200). Then for that choice of $r$, plot all those values (i.e., $x_{n_S}$ to $x_{n_S+N}$). You should end up with something that has a similar shape to what is shown in Fig.1 for the quadratic map.

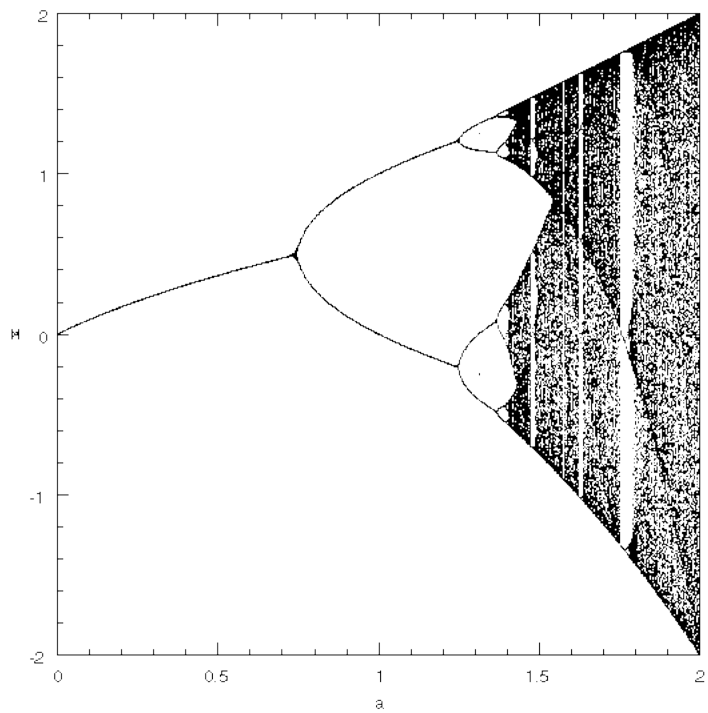


Figure 1: Period doubling cascade for for the quadratic map.

b. Briefly describe what the ‘period doubling cascade’ means.

c. Repeat a similar computation for the quadratic map.

d. Briefly describe how these systems are related to problems in biology. [Hint: See ‘Simple mathematical models with very complicated dynamics’ (Nature, 1976) by Robert May]

**Extra Credit**: Based upon your framework above, go on step further and develop a program to create a fractal based off the *Julia set*.