Modeling Cochlear Dynamics

Christopher Bergevin

Department of Mathematics
University of Arizona

10/18/07
Disclaimer: \( j \equiv \sqrt{-1} = i \)
Inner ear (cochlea, fluid-filled)

Middle ear (ossicles, air-filled)

Outer ear (pinna, ear canal & ear drum)
Inner ear - Organ of Corti

- Three fluid filled compartments
- Sensory complex sits atop a flexible membrane *(basilar membrane)*
Georg von Bekesy (1899-1972)
(Nobel Prize in 1961)
Fig. 11-59. Detail of the form of vibration of the cochlear partition for 200 cps at two instants within a cycle.

Fig. 11-58. Phase displacement and resonance curves for four low tones.
**Question**: Can we develop a model based upon the anatomy that captures the observed physiological features?

**Goal**: Model should serve as a foundation

⇒ 1-D transmission-line model solved using WKB approximation
Assumptions

- effect of coiling is negligible
  (allows us to ‘unroll’ the cochlea)

- height of traveling wave is small relative to the height of the scalae
  (pressure is uniform in both cross-sections and depends only upon longitudinal distance)

- fluid is incompressible and viscosity negligible
Geometry

$x$ - longitudinal distance along cochlea

$b$ and $h$ - width and height of the scalae (assumed constant for now)

$d$ - vertical (transverse) displacement of the BM
**Diagram Description**

- **Stapes**
- **Round Window**
- **Scala Vestibuli**
- **Scala Tympani**
- **Area A** (width = b, height = h)
- **p** - pressure
- **u** - fluid velocity
- **ρ** - fluid density

**Mathematical Elements**

- $\rho_v$
- $u_v$
- $p_t$
- $u_t$
- Vertical displacement = $d$

**Coordinates**

- $x$
- $x + \Delta x$
Fluid Flow Due to Pressure Difference

Consider element of scala vestibuli. Using Newton’s 2nd law (no fluid viscosity):

\[
\Delta t \cdot A_v \cdot [p_v(x, t) - p_v(x + \Delta x, t)] = \rho \cdot \Delta x \cdot A_v \cdot [u_v(x, t + \Delta t) - u_v(x, t)]
\]

\[
\frac{\partial p_v}{\partial x} = -\rho \frac{\partial u_v}{\partial t}
\]

\[
\frac{\partial p_t}{\partial x} = -\rho \frac{\partial u_t}{\partial t}
\]
Fluid Velocity to Membrane Displacement

From the conservation of mass (incompressible fluid):

\[ \Delta t \cdot A_v \cdot \rho \cdot [u(x, t) - u(x + \Delta x, t)] = \Delta x \cdot \rho \cdot b \cdot [d(x, t + \Delta t) - d(x, t)] \]

\[ A_v \frac{\partial u_v}{\partial x} = -b \frac{\partial d}{\partial t} \]

\[ A_t \frac{\partial u_t}{\partial x} = b \frac{\partial d}{\partial t} \]
BM Motion Due to Pressure Difference

Consider all forces acting on the BM:

\[ F = F_{\text{stiffness}} + F_{\text{drag}} + F_{\text{pressure}} = \mu \cdot \Delta x \cdot \frac{\delta^2 d}{\delta t^2} \]

\[ b \cdot [p_v(x, t) - p_t(x, t)] = \mu \cdot \frac{\delta^2 d(x, t)}{\delta t^2} + \alpha \cdot \frac{\delta d(x, t)}{\delta t} + \kappa \cdot d(x, t) \]
Some consequences

\[ u_v(x, t) = -u_t(x, t) \]

\[ p_v(x, t) + p_t(x, t) = \alpha \]
Simplifications

Let: \( p \equiv p_v - p_t \quad u \equiv A_{cs}(u_v - u_t)/2 \)

Assume both \( u \) and \( p \) are in sinusoidal steady-state (stimulus frequency \( \omega \)) such that:

\[
\begin{align*}
    p(x, t) &= \Re[P(x, \omega)e^{i\omega t}] \\
    u(x, t) &= \Re[U(x, \omega)e^{i\omega t}]
\end{align*}
\]

\[
\frac{\partial P}{\partial x} = -\frac{2\rho}{A_{cs}}i\omega U = -ZU \quad (\text{eqn. I})
\]
Simplifications II

\[
\frac{\partial p}{\partial x} = \frac{A_{cs}}{2} \left[ -\frac{b}{A_{cs}} \left( \frac{\partial d}{\partial t} + \frac{\partial d}{\partial t} \right) \right] = -b \frac{\partial d}{\partial t}
\]

\[
d(t) = \int -\frac{e^{j\omega t}}{b} \frac{\partial U}{\partial x} dt = -\frac{e^{j\omega t}}{j\omega b} \frac{\partial U}{\partial x}
\]

Plugging back into the equation of motion (relating pressure and displacement):

\[
\frac{\partial U}{\partial x} = -\frac{P}{j\omega \frac{\mu}{b^2} + \frac{\alpha}{b^2} + \frac{1}{j\omega \frac{b^2}{\kappa}}} = -YP 
\]

(eqn. II)
Wave Equation

\[ \frac{\partial P}{\partial x} = -\frac{2\rho}{A_{cs}} i\omega U = -ZU \]

\[ \frac{\partial U}{\partial x} = - \frac{P}{j\omega \frac{\mu}{b^2} + \frac{\alpha}{b^2} + \frac{1}{j\omega \frac{b^2}{\kappa}}} = -YP \]

\[ \frac{\partial^2 P}{\partial x^2} + \frac{1}{\ell^2} P = 0 \]

\[ \ell = j\sqrt{1/ZY} \]

\[ \frac{\partial^2 U}{\partial x^2} + \frac{1}{\ell^2} U = 0 \]

\[ p(x, t) = P_1 e^{i(x/\ell + \omega t)} + P_2 e^{i(-x/\ell + \omega t)} \]
Analogy to Electrical Transmission Line

**Electrical Case** (loss-less)

\[ -\frac{\partial v}{\partial x} = L \frac{\partial i}{\partial t} \]

\[ -\frac{\partial i}{\partial x} = C \frac{\partial v}{\partial t} \]

**Cochlear Case**

\[ \frac{\partial P}{\partial x} = -ZU \]

\[ \frac{\partial U}{\partial x} = -YP \]
Assumptions Revisited

$b, A$ not constant  

$Z \rightarrow Z(x)$  

$Y \rightarrow Y(x)$  

$\Rightarrow$ non-uniform transmission line
WKB Approximation

Wentzel, Kramers and Brillouin (1926)

Particle with energy $E$ moving in constant potential field $V$:

$$\psi(x) = A e^{\pm j k x} \quad \quad \quad k \equiv \sqrt{2m(E - V)/\hbar}$$

⇒ What if $V$ is not constant, but varies gradually?

$$\psi(x) \approx \frac{C}{\sqrt{k(x)}} e^{\frac{i}{\hbar} \int k(x) dx} \quad \quad \quad k(x) \equiv \sqrt{2m[E - V(x)]/\hbar}$$
WKB Applied to Cochlea

Assumption: Cochlear parameters (e.g. BM stiffness, scalae area) vary *gradually* such that cochlea behaves like a uniform transmission line *locally*

\[
P(x, \omega) = A(x) e^{-j \int_0^x dx' / \ell(x', \omega)}
\]

\[
\ell(x, \omega) = j \left[ \frac{j \omega L + R + 1/j \omega C}{j \omega M} \right]^{1/2}
\]

\[
\ell(x, \omega) = \frac{l}{4N} \frac{(1 - \beta^2 + j \delta \beta)^{1/2}}{\beta}
\]

\[
\omega_r(x) \equiv 1/\sqrt{LC} \quad \beta(x, \omega) \equiv \omega/\omega_r(x) \quad \delta \equiv \omega_r(x) RC \quad N \equiv (l/4)\sqrt{M/L}
\]
Transfer Function

\[ T(x, \omega) = \frac{\text{BM velocity}}{\text{stapes velocity}} \]

BM velocity = \( \frac{e^{j\omega t}}{b} \ell^2(x, \omega) Y e^{-j \int_0^x dx'/\ell(x', \omega)} \)

\[ \omega_r(x) = \omega_{max} e^{-x/l} \]
WKB Applied to Cochlea (cont.)

\[ \beta = \omega / \omega_r \quad \text{and} \quad dx = \ell \frac{d\beta}{\beta} \]

\[ 4N \int_0^x \frac{d\beta}{[1 - \beta^2 + j\delta\beta]^{1/2}} \]

\[ T(x, \omega) \approx T_0 j\beta(x, \omega) \left[ \frac{\omega_{\text{max}}}{\omega_r(x)} \right] e^{j4N \left[ \sin^{-1} \left( \beta(x, \omega) - j\delta/2 \right) - \sin^{-1} \left( \beta(0, \omega) - j\delta/2 \right) \right]} \frac{1}{\left[ 1 - \beta^2(x, \omega) + j\delta\beta(x, \omega) \right]^{3/4}} \]

(Zweig, 1991)
Comparison to Data

Physiological measurements of BM motion relative to stapes displacement

[Fig. 11-58. Phase displacement and resonance curves for four low tones.]

[Rhode, 1971]
Model provides a starting point for thinking about cochlear dynamics

⇒ What features are present in a real ear that we would like the model to capture?
I. Near Threshold in Viable Ears

Model can not capture sharp tuning/large group delay at CF

⇒ Need for active mechanisms? [Neely and Kim, 1983]
II. Nonlinearity

Compressive BM Growth (on CF)

BM response at a fixed spot to different frequency tones

\[ \text{Velocity (\mu m/s)} \]

\[ \text{Sound Pressure Level (dB)} \]

\[ \text{CF = 10 kHz} \]

\[ \text{slope = 1 dB/dB e.g. linear growth} \]

from Ruggero et al. (JASA, 1997)

Two-tone Distortion in Pressure

\[ f_2 / f_1 = 1.05 \]

\[ L_1 = L_2 = 60 \text{ dB SPL} \]

\[ f_2 / f_1 = 1.05 \]

\[ L_1 = L_2 = 80 \text{ dB SPL} \]

\[ f_1 \]

\[ f_2 \]

\[ f_1 \]

\[ f_2 \]

\[ f_1 \]

\[ f_2 \]

\[ f_1 \]

\[ f_2 \]

\[ f_1 \]

\[ f_2 \]

Dong and Olson (JASA, 2005)

⇒ Source of nonlinearity?
III. OAEs

**Otoacoustic Emissions (OAEs)**
sounds emitted by a normal, healthy ear

⇒ Model could serve to elucidate OAE generation
Reflection of energy?

III. OAEs (cont.)

Transmission line with random irregular ‘sources’

Shera and Guinan (2007)
Summary

Developed a passive, linear 1-D transmission line model for the cochlea

Simple model captures essential features of the cochlea and can serve as a foundation for more realistic iterations

Ultimate goal is to use cochlear models to better understand auditory function/physiology and potential clinical applications
Georg von Bekesy (1899-1972)

Fig. 11-48. Method of measuring the amplitude of vibration of the cochlear partition in response to volume displacements of the stapes.