COV VS. PP SUPPLEMENT

This proof comes from *Cardinal Arithmetic*, p. 88.

Theorem 1. *Suppose σ is regular and uncountable, and $\theta > \sigma$. Then*

$$\sup\{\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda^*) : \lambda^* \in [\kappa, \lambda], \sigma \leq \mathrm{cf}(\lambda^*) < \theta)\} + \lambda = \mathrm{cov}(\lambda, \kappa, \theta, \sigma) + \lambda.$$

Proof. In class we proved the "$\leq$" inequality and saw the following two exercises:

Exercise 2. If $\lambda > \kappa$ and σ is regular, then

$$\mathrm{cov}(\lambda, \kappa, \theta, \sigma) = \sum_{\mu \in [\kappa, \lambda]} \mathrm{cov}(\mu, \mu, \theta, \sigma).$$

Exercise 3. If $\lambda = \kappa$ and $\mathrm{cf}(\lambda) < \sigma$ then $\mathrm{cov}(\lambda, \kappa, \theta, \sigma) \leq \lambda$.

So we may assume that $\lambda = \kappa$ and $\mathrm{cf}(\kappa) \geq \sigma$.

Suppose $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda) = \mu > \lambda$ (things reduce somewhat under our assumptions on the cardinals). We will prove that $\mathrm{cov}(\lambda, \kappa, \theta, \sigma) \leq \mu$.

Let $\chi = (2^{2^\lambda})^+$ and choose $N \prec (H_\chi, \in, <^*)$ so that $|N| = \mu$ and $\mu + 1 \subseteq N$. We will prove that $[\lambda]^{<\kappa} \cap N$ is a $< \sigma$-covering family.

For this, suppose $\sigma \leq \theta^* < \theta$ and $f : \theta^* \to \lambda$ is the increasing enumeration of a set to be covered.

Define I to be the σ-complete ideal of all $B \subseteq \theta^*$ for which $f``B$ is a $< \sigma$-sized union of members of $[\lambda]^{<\kappa} \cap N$. In case f cannot be covered, I is a proper ideal. Let

$$H = \{h : h : \theta^* \to P(\lambda) \cap N, (\forall i < \theta^*)\, f(i) \in h(i),\ \text{and there exist } \xi^* < \sigma$$
$$\text{and } \langle X_\zeta : \zeta < \zeta^* \rangle \text{ so that } X_\zeta \in [P(\lambda)]^{<\kappa} \cap N, \mathrm{range}(h) \subseteq \bigcup_{\zeta < \zeta^*} X_\zeta\}.$$

Let G be the set of functions with domain θ^* so that there is some $h \in H$ for which $g(i) = |h(i)|$. The sets H, G are nonempty—the constant $\{\lambda\}$ function is in H, for example, corresponding to the constant λ function in G.

Let $g^* \in G$ be $<_I$-minimal, corresponding to some $h^* \in H$ witnessed by some $\langle X_j : j < j^* \rangle$.

Claim 4. *The set $\{i < \theta^* : g^*(i) = 1\}$ belongs to I.*

Proof of claim: for each $j < j^*$, the set $\{\alpha < \lambda : \{\alpha\} \in X_j\} \in [\lambda]^{<\kappa} \cap N$, so the claim follows immediately from the definition of I by noting that the image of $\{i < \theta^* : g^*(i) = 1\}$ under f is the union over j^* of such sets. $\dashv$

Take in N a function which associates to $y \subseteq \lambda$ a $\subseteq$-increasing sequence $\langle y^{[\epsilon]} : \epsilon < \mathrm{cf}(|y|) \rangle$ so that $y = \bigcup_{\epsilon < \mathrm{cf}(|y|)} y^{[\epsilon]}$.

Let $\lambda_i := \mathrm{cf}(g^*(i))$.

Let $\tau^* = \sup_j |X_j|$. Then $\tau^* < \kappa$ since $\kappa = \lambda$ has cofinality $\geq \sigma > j^*$. For every i with $g^*(i) > \tau^*$, there is some $j < j^*$ so that

$$\lambda_i := \mathrm{cf}(g^*(i)) \in \{\mathrm{cf}(|y|) : y \in X_j, \mathrm{cf}(|y|) > |X_j|\} =: \mathfrak{b}_\mathrm{j}.$$

The point is that for each $j < j^*$, $\mathfrak{b}_\mathrm{j} \in N$, even though $\{\lambda_i : i < \theta^*\}$ may not be. There are $< \sigma$ many such $\mathfrak{b}_\mathrm{j}$. In N, fix pcf generators $B^j_\tau := B_\tau[\mathfrak{b}_j]$, $\tau < \mu$. By (a J^{bd}_λ version of) the characterizations of $J^{\sigma\text{-com}}_{<\mu}$ we proved and the assumption that $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda) = \mu$, for each $j < j^*$, there is some $\kappa_j < \kappa$ so that $\{\lambda_i : i < \theta^*\} \cap \mathfrak{b}_j \setminus \kappa_j$ is a subset of the union of $< \sigma$ many of the B^j_τ, $\tau \leq \mu$. Now $B^j_\tau \in N$ since $\tau \leq \mu$ so B^j_τ in N. We can work on each B^j_τ individually to attempt to reduce the values of g^*.

So fix B^j_τ used above. By the calculation of cofinality of products, we can pick a cofinal family $\mathcal{P}^j_\tau \in N$ in $(\prod B^j_\tau, <)$ so that $|\mathcal{P}^j_\tau| = \tau$ and hence $\mathcal{P}^j$ is a subset of N.

We define a function $e \in \prod_{i:\lambda_i \in B^j_\tau} \lambda_i$ by mapping i to the least ϵ so that $f(i) \in g(i)^{[\epsilon]}$. Now there is $e^j_\tau \in \mathcal{P}_j$ so that $e < e^j_\tau$.

Define X^τ_j to be

$$X_j \cup \{y^{[e^j_\tau(\mathrm{cf}(|y|))]} : y \in X_j, \mathrm{cf}(|y|) \in \mathfrak{b}_j\}.$$

The family X^τ_j is in N and has size $< \kappa$.

We will now define h^{**} in H. If $\lambda_i = 1$, let $h^{**}(i) = h^*(i)$. If $\lambda_i \in B^j_\tau$ for some j,τ, define $h^{**}(i)$ to be $h^*(i)^{[e^j_\tau(\lambda_i)]}$ for the lexicographically least such (j, τ).

Let A be the set of i for which neither of the two cases above holds, consisting of the indices of the bounded set of λ_i thrown out for each $j < j^*$ in the construction above. Since $\mathrm{cf}(\kappa) \geq \sigma$, there is a $\kappa^* < \kappa$ so that $\lambda_i < \kappa^*$ for all $i \in A$. We can expand X^τ_j for each τ, j so that for every $y \in X^\tau_j$, if $\mathrm{cf}(|y|) \leq \kappa^*$ then $y^{[\epsilon]} \in X^\tau_j$. Since $\kappa^* < \kappa$, adding all of these sets $y^{[\epsilon]}$ to X_j keeps the cardinality of X^τ_j below κ (and we get the desired X^τ_j by iterating this procedure ω times, in N). Now for $i \in A$ define $h^{**}(i) = h^*(i)^{[\epsilon(i)]}$, where $\epsilon(i) < \lambda_i$ is minimal with $f(i) \in h^*(i)^{[\epsilon(i)]}$.

The collection of all X^τ_j ($j < j^*$ and τ taken as above) is of size $< \sigma$ and witnesses that $h^{**} \in H$. However, the associated g^{**} is $<_I$-below g^*, contradicting minimality of g^*. $\square$