# ANTICHAINS, THE STICK PRINCIPLE, AND A MATCHING NUMBER

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ABSTRACT. We investigate basic properties of three cardinal invariants involving $\omega_1$: stick, antichain number, and matching number. The antichain number is the least cardinal $\kappa$ for which there does not exist a subcollection of size $\kappa$ of $[\omega_1]^{\omega_1}$ with pairwise finite intersections and the matching number is the least cardinal $\kappa$ for which there exists a subcollection $X$ of size $\kappa$ of order-type $\omega$ subsets of $\omega_1$ so that every uncountable subset of $\omega_1$ has infinite intersection with a member of $X$. We demonstrate how these numbers are affected by Cohen forcing and also prove some results about the effect of Hechler forcing. We also introduce a forcing notion to increase the matching number, and study its basic properties.

## 1. INTRODUCTION

In this paper, we study the connection between antichains of $[\omega_1]^{\omega_1}$ modulo finite and the stick number $\stick$. For $X$ a set of ordinals and $\delta$ an ordinal, we use the notation $[X]^\delta$ to denote the collection of subsets of $X$ of order-type $\delta$. For $\delta \in [\omega, \omega_1)$, define

$$\stick_\delta = \min\{|X| : X \subseteq [\omega_1]^\delta \text{ and } \forall y \in [\omega_1]^{\omega_1} \exists x \in X (x \subseteq y)\}. \tag{1}$$

We will denote $\stick_\omega$ simply by $\stick$.

It is easy to verify that $\aleph_1 \leq \stick \leq 2^{\aleph_0}$, so $\stick$ assumes values typical of the cardinal characteristics of the continuum. In fact, there are some known relationships between $\stick$ and well-studied cardinal characteristics (see Brendle [5]). To point out an early example, Truss [14] showed that $\stick$ is at least the minimum of the covering number of the meagre ideal and the covering number of the Lebesgue-null ideal.[1]

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[1] More precisely, Truss showed that the two-step iteration of adding a Cohen real followed by adding a random real adds an uncountable subset of $\omega_1$ which does not contain any infinite subset from the ground model; in modern language, $\mathsf{Cohen} * \mathsf{Random}$ is not $(\omega, \omega_1)$-semidistributive. To show the inequality of cardinals, take an elementary submodel $M$ of a sufficiently large initial segment of the universe so that $|M| = \stick$ and contains each member of the family witnessing $\stick$. If the covering numbers of the null and meager ideals are both larger than $\stick$, then we can find a $c \in V$ Cohen-generic over $M$ and $r \in V$ random-generic over $M[c]$, using the measure and category characterizations of genericity. Now in $M[c * r]$ there is an an uncountable subset of $\omega_1$ which does not contain any infinite subset from the ground model, contradicting that $M$ contains each member of a family witnessing $\stick$.

We also study another cardinal invariant that reflects the combinatorics at $\omega_1$ and is such that every smaller cardinal lies in the range between $\omega_1$ and the continuum. Define the antichain number $\mathfrak{cc}(\kappa, \nu)$ to be the least $\lambda$ for which there does not exist a collection of $\lambda$-many subsets of $\kappa$ of cardinality $\kappa$ with pairwise intersections of size $< \nu$. We will refer such a collection as an *antichain*. We are also interested in $\mathfrak{cc}(\omega_1, \omega)^-$, where the superscript $^-$ means predecessor if the original cardinal happens to be successor. This can also be defined as the supremum of the cardinalities of antichains. With some additional parameters, these quantities were studied by Baumgartner [2].

It is not difficult to see that $\mathfrak{cc}(\kappa, \nu)$ is always a regular cardinal. A standard argument gives the relationship with stick.

**Fact 1.1.** If $\lambda < \mathfrak{cc}(\omega_1, \omega)$, then $\lambda \leq$ ⫯. In other words, $\mathfrak{cc}(\omega_1, \omega)^- \leq$ ⫯.

One way to increase ⫯ and similar invariants—witnessed by an object with absolute properties—is to construct large antichains of the relevant type. Historically, this method was often used; see Baumgartner [2], Kojman [12], or Chen [7].

A third notion seems fundamental: the matching number, introduced in Section 2. In the simplest case, $\mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega)$ is the smallest cardinality of a collection of order-type $\omega$ subsets of $\omega_1$ so that for any uncountable subset of $\omega_1$, there is a member of the collection which it intersects infinitely. By adjusting the values of the parameters, this definition gives rise to many interesting cardinal invariants.

The present work aims to extend the study of cardinal characteristics to these numbers which involve combinatorics at $\omega_1$. In particular, we examine the possible values they can take in forcing extensions by various posets—Cohen (Section 4), Hechler (Section 5), and a natural forcing designed to influence the matching number (Section 3). In some cases we manage only to obtain partial results; however, these partial results lie along interesting dividing lines and offer a starting point for future investigation.

In Section 6, we consider a variation ⫯$^{\mathrm{ad}}$ of ⫯ where the witnessing family is required to be pairwise almost disjoint and show that this number is equal to ⫯ in all cases, answering a question of Galgon [8].

The notation we use is largely standard. If $x$ is a set of ordinals, then $\mathrm{ot}(x)$ denotes its order-type and $\mathrm{Lim}(x)$ denotes the set of its limit points, i.e., $\{\alpha : \sup(x \cap \alpha) = \alpha\}$. When $\alpha$ is an ordinal, we also use the notation $\mathrm{ot}(\alpha)$ to refer to the property of having order-type $\alpha$, and the notation $\lim(\alpha)$ to denote the set of limit ordinals less than $\alpha$. $\mathsf{MA}$ refers to the version of Martin's Axiom that asserts the existence of filters which meet $< 2^{\aleph_0}$ given dense subsets of a c.c.c. poset.

The cardinals $\mathfrak{b}$ and $\mathfrak{d}$ play an important role in this paper. Let ${}^\omega\omega$ be the collection of functions $f : \omega \to \omega$, and let $<^*$ be the order defined on ${}^\omega\omega$ to be $f <^* g$ if $|\{\alpha < \omega : g(n) \leq f(n)\}| < \omega$. The *unbounding number* $\mathfrak{b}$ is the least cardinality of a family $A \subseteq {}^\omega\omega$ so that for any $f \in {}^\omega\omega$ there is a $g \in A$ so that $g \not<^* f$. The *dominating number* $\mathfrak{d}$ is the least cardinality of a family $A \subseteq {}^\omega\omega$ so that for any $f \in {}^\omega\omega$ there is a $g \in A$ so that $f <^* g$.

Following Hrušák [9], for $\kappa \leq \lambda$ say that a model $M$ is $(\kappa, \lambda)$-semidistributive over $V \subseteq M$ if every subset of $\lambda$ of size $\lambda$ in $M$ contains a subset of size $\kappa$ in $V$.

## 2. The matching number

We describe a weakening of the stick number $\text{stick}$ which we denote $\mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega)$, the smallest cardinality of a collection of $\mathrm{ot}(\omega)$ subsets of $\omega_1$ which is countably matching for uncountable subsets of $\omega_1$.

If we weaken the guessing condition in 1 from $x \subseteq y$ to the matching condition

$$|x \cap y| = \omega$$

and strengthen the requirement that the elements of $x$ must be countable to insisting that they must be of order-type $\omega$, we arrive at $\mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega)$. That is, $\mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega)$ is defined to be the minimal cardinality of a set $X \subseteq [\omega_1]^\omega$ such that for every $y \in [\omega_1]^{\omega_1}$, there exists $x \in X$ with $|x \cap y| = \omega$. It is clear that $\omega_1 \leq \mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega) \leq \text{stick}$. More generally we may define:

**Definition 2.1.** For $\gamma \leq \delta \leq \kappa \leq \lambda$, $\mathfrak{mat}_\delta([\lambda]^\kappa, \gamma)$ is the minimal cardinality of an $X \subseteq [\lambda]^\delta$ such that for every $y \in [\lambda]^\kappa$ there exists $x \in X$ with $\mathrm{ot}(x \cap y) \geq \gamma$.

**Fact 2.2.** The quantity $\mathfrak{mat}_\delta([\lambda]^\kappa, \gamma)$ is non-increasing in the parameters $\kappa$ and $\delta$ and non-decreasing in $\gamma$ and $\lambda$.

If the elements of a matching family are limited by cardinality instead of order-type, we can also consider natural variants like $\mathfrak{mat}_{<\mu}([\lambda]^\kappa, \gamma)$, and others. In the next section, we will show to how increase the value of $\mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega)$ by forcing.

Juhász considers a weakening of ♣ named the principle $(t)$ in [11] and argues that $(t)$ holds after adding a Cohen real to a model of ZFC. A more direct analogue to the matching numbers we are considering here is the natural ♣-like weakening of $(t)$ considered by D. Soukup [13] and called the weak $(t)$ axiom, that is that there exists a sequence $\langle A_\alpha : \alpha \in \lim(\omega_1)\rangle$ with $A_\alpha \in [\alpha]^\omega$ cofinal such that for every $X \in [\omega_1]^{\omega_1}$, there exists $\alpha \in \lim(\omega_1)$ with $|X \cap A_\alpha| = \omega$. Soukup notes that the weak $(t)$ axiom fails under MA, but holds after adding a single Cohen real. We can observe using a different method that it holds after adding a single real of a broader class.

**Theorem 2.3.** *If $M$ is $(\omega, \omega_1)$-semidistributive over $V$ and contains an unbounded real, then $\mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega) = \omega_1$ in $M$.*

*Proof.* Let $V \subseteq M$ be as in the hypothesis, and let $f \in {}^\omega\omega$ be unbounded and monotone over $V$. Working in $V$, for every $\alpha \in \lim(\omega_1)$ larger than $\omega$, fix $\langle \xi_n : n < \omega\rangle \subseteq \alpha$ a cofinal strictly increasing sequence of ordinals such that $|\xi_{n+1} \setminus \xi_n| = \omega$ for every $n < \omega$. Enumerate $\xi_{n+1} \setminus \xi_n = \{\beta^n_k : k < \omega\}$ in type $\omega$. In $M$, let $A_\alpha$ comprise the first $(f(n)+1)$-many $\beta$'s in each interval. That is, $A_\alpha = \bigcup_{n<\omega} \{\beta^n_k : k \leq f(n)\}$. Let $A_\omega = \omega$.

Now, let $x \in [\omega_1]^{\omega_1}$. There exists $y \in ([\omega_1]^\omega)^V$ such that $y \subseteq x$. First, find the unique $\alpha$ such that $y \cap \alpha \in [\alpha]^\omega$. If $\alpha = \omega$ then $|y \cap A_\omega| = \omega$, so suppose $\alpha > \omega$. Working in $V$, let $z = \{n < \omega : y \cap (\xi_{n+1} \setminus \xi_n) \neq \emptyset\}$. Let $g_y \in {}^z\omega$ be defined by setting $g_y(n)$ to be the unique $m$ such that $\inf\{(\xi_{n+1} \setminus \xi_n) \cap y\} = \beta^n_m$. Let $g \in {}^\omega\omega$ be defined as $g_y \circ z$, that is $g(k) = g_y(z(k))$ for every $k < \omega$, where we identify $z$ with its enumerating function. Because $f$ is unbounded, there exists infinitely many $k$ with $f(k) \geq g(k)$, and because $f$ is monotone, $f(z(k)) \geq f(k) \geq g_y(z(k))$. But then $A_\alpha$ intersects $y$ in infinitely-many intervals, so $|A_\alpha \cap y| = \omega$. □

*Remark* 2.4. Note that Cohen forcing is $(\omega_1, \omega_1)$-semidistributive as a result of being countable and adds unbounded reals, so satisfies the hypotheses of Theorem 2.3. However, Laver forcing for example does not add Cohen reals but also satisfies the hypotheses of Theorem 2.3.

*Remark* 2.5. Juhász's observation can be used to show that consistently, the matching number $\mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega)$ is less than the antichain number $\mathfrak{cc}(\omega_1, \omega)^-$: simply increase $\mathfrak{cc}(\omega_1, \omega)^-$ by any means, and then add a Cohen real (or any use any forcing satisfying the hypotheses of Theorem 2.3).

The argument in Theorem 2.3 can also show that matching families for $[\lambda]^\kappa$ consisting of elements of order-type $\kappa$ have a connection to the (un)bounding number $\mathfrak{b}(\kappa)$. Similarly, covering collections for $[\lambda]^\kappa$ consisting of elements of order-type $\kappa$ have a connection to the dominating number $\mathfrak{d}(\kappa)$. To state these connections, it is helpful to define the covering number. For $\gamma \leq \delta \leq \lambda$ define $\mathfrak{cov}_\delta([\lambda]^\gamma)$ to be the minimal cardinality of a covering set for $[\lambda]^\gamma$ consisting of $\mathrm{ot}(\delta)$ subsets of $\lambda$. That is, the minimal cardinality of an $X \subseteq [\lambda]^\delta$ such that for every $y \in [\lambda]^\gamma$, there exists $x \in X$ with $y \subseteq x$. The following Proposition 2.6 is straightforward to prove using the method of Theorem 2.3 along with an induction argument.

**Proposition 2.6.** *If $\kappa$ is regular and $\lambda \in (\kappa, \aleph_\kappa)$, then $\mathfrak{mat}_\kappa([\lambda]^\kappa, \kappa) = \lambda \cdot \mathfrak{b}(\kappa)$ and $\mathfrak{cov}_\kappa([\lambda]^\kappa) = \lambda \cdot \mathfrak{d}(\kappa)$.*

*Remark* 2.7. Proposition 2.6 says that $\mathfrak{b}$ is the minimal cardinality of a collection of $\mathrm{ot}(\omega)$ subsets of $\omega_1$ which is matching for $[\omega_1]^\omega$ and $\mathfrak{d}$ is the minimal cardinality of a collection of $\mathrm{ot}(\omega)$ subsets of $\omega_1$ which is covering for $[\omega_1]^\omega$.

This connection between $\mathrm{ot}(\omega)$-matching and covering, and unbounded and dominating reals can also be phrased in terms of models.

**Definition 2.8.** Let $V \subseteq M$ be models. Then $V$ is $([\omega_1]^\omega, \mathrm{ot}(\omega))$-*covering* in $M$ if for every $y \in ([\omega_1]^\omega)^M$, there exists $x \in ([\omega_1]^\omega)^V$ with $y \subseteq x$. $V$ is $([\omega_1]^\omega, \mathrm{ot}(\omega))$-*matching* in $M$ if for every $y \in ([\omega_1]^\omega)^M$, there exists $x \in ([\omega_1]^\omega)^V$ with $|x \cap y| = \omega$.

**Proposition 2.9.** *Let $V \subseteq M$ be models. Then $V$ is $([\omega_1]^\omega, ot(\omega))$-covering in $M$ if and only if $M$ does not add an unbounded real. Similarly, $V$ is $([\omega_1]^\omega, ot(\omega))$-matching in $M$ if and only if $M$ does not add a dominating real.*

It is useful to have notation for the type of subsets witnessing e.g. a failure of matching or covering as we have been considering, just as one can refer to an unbounded real as witnessing the failure of ${}^\omega\omega$-bounding. The following notation is suitable, and can be easily generalized to other ordinal order-types, disjointing modulo sets of cardinality less than $\kappa$, etc.

**Definition 2.10.** If $V \subseteq M$ are models, say that $M$ contains an $[\omega_1]^\omega$-disjointing $\mathrm{ot}(\omega)$-subset over $V$ if there exists some $x \in ([\omega_1]^\omega)^M$ such that for every $y \in ([\omega_1]^\omega)^V$, $|x \cap y| < \omega$. Similarly, say that $M$ contains an $[\omega_1]^\omega$-uncovered $\mathrm{ot}(\omega)$-subset if and only if there exists $x \in ([\omega_1]^\omega)^M$ such that for every $y \in ([\omega_1]^\omega)^V$, $\neg(x \subseteq y)$.

With the terminology of Definition 2.10, Proposition 2.9 says that a dominating real is added if and only if an $[\omega_1]^\omega$-disjointing $\mathrm{ot}(\omega)$-subset is added and an unbounded real is added if and only if an $[\omega_1]^\omega$-uncovered $\mathrm{ot}(\omega)$-subset is added.

## 3. The poset $\mathbb{P}_{\mathfrak{mat}_\delta}$

In this section, we investigate a natural forcing notion to increase the matching number.

**Definition 3.1.** Let $\mathbb{P}_{\mathfrak{mat}_\delta}$ consist of conditions $p = (s_p, c_p)$ where $s_p \in [\omega_1]^{<\omega}$ and $c_p \in [\omega_1]^{<\delta\omega}$, $\mathrm{ot}(c_p) = \delta k$ for some $k < \omega$. Say that $(s_q, c_q) \leq (s_p, c_p)$ if and only if $s_q \supseteq s_p$, $c_q \supseteq c_p$, and $(s_q \setminus s_p) \cap c_p = \emptyset$.

It is not difficult to see that forcing with $\mathbb{P}_{\mathfrak{mat}_\delta}$ adds a subset $A_G = \bigcup_{p \in G} s_p$ such that $|A_G| = |\omega_1^V|$ and $|A_G \cap a| < \omega$ for every $a \in ([\omega_1]^\delta)^V$. We will check that $\mathbb{P}$ is c.c.c.

**Theorem 3.2.** *The poset $\mathbb{P}_{\mathfrak{mat}_\delta}$ is c.c.c.*

*Proof.* Let $\{p_\xi = (s_\xi, c_\xi) : \xi < \omega_1\} \subseteq \mathbb{P}_{\mathfrak{mat}_\delta}$. Using the $\Delta$-system lemma, we may assume that $\{s_\xi : \xi < \omega_1\}$ forms a $\Delta$-system with root $r$. Note that for $\xi < \omega_1$, there exist at most countably-many $\xi' < \omega_1$ such that $(s_{\xi'} \setminus r) \cap c_\xi \neq \emptyset$. So by iteratively removing offending conditions, we may assume that for $\xi < \xi' < \omega_1$, $(s_{\xi'} \setminus r) \cap c_\xi = \emptyset$. By a further thinning, we may similarly assume that for $\xi < \xi' \in \omega_1$, $\sup(r) < \min(s_\xi \setminus r)$ and $\sup(s_\xi) < \inf(s_{\xi'} \setminus r)$.

For future reference, we will call a $\Delta$-system $\{s_\xi : \xi < \omega_1\}$ *head-tail-tail* with root $r$ if it satisfies the conditions of the previous sentence.

Next, consider $p_{\delta\omega}$. Enumerate $c_{\delta\omega} = \langle \alpha_\beta : \beta < \mathrm{ot}(c_{\delta\omega}) \rangle$ in increasing order. For any $\xi < \delta\omega$, because $(s_{\delta\omega} \setminus r) \cap c_\xi = \emptyset$, if $p_\xi \perp p_{\delta\omega}$ then necessarily for some $\beta_\xi < \delta k$, $\alpha_{\beta_\xi} \in s_\xi$. Suppose towards a contradiction that $p_\xi \perp p_{\delta\omega}$ for every $\xi \in \delta\omega$. Then $\langle \alpha_{\beta_\xi} : \xi < \delta\omega \rangle \subseteq \langle \alpha_\beta : \beta < \mathrm{ot}(c_{\delta\omega}) \rangle$ is increasing, contradicting $\mathrm{ot}(c_{\delta\omega}) < \delta\omega$. $\square$

This forcing can be iterated to increase the matching number. For example,

**Corollary 3.3.** *Under* MA*, we have $\mathfrak{mat}_\delta([\omega_1]^{\omega_1}, \omega) = \mathfrak{mat}_\delta([\omega_1]^{\delta\omega}, \omega) = 2^\omega$ for every $\delta \in [\omega, \omega_1)$.*

We observe that the $\mathbb{P}_{\mathfrak{mat}_\delta}$ forcings can consistently add a large antichain.

**Proposition 3.4.** *There is a model $V$ of $\stick = \aleph_1$ so that $\mathfrak{cc}(\omega_1, \omega) > \omega_1$ in the extension obtained by forcing with the poset $\mathbb{P}_{\mathfrak{mat}_{\omega^\omega}}$ over $V$.*

*Proof.* Start from the model $V$ of Chen [7] where $\stick = \aleph_1$ and there is a collection of $\aleph_2$-many subsets $\langle A_\alpha : \alpha < \omega_2 \rangle$, $A_\alpha \subseteq \omega_1$ whose pairwise intersections have order-type less than $\omega^\omega$. Let $e_\alpha : \omega_1 \to A_\alpha$ be the increasing enumeration of $A_\alpha$ for each $\alpha < \omega_2$. Note that $\mathbb{P}_{\mathfrak{mat}_{\omega^\omega}}$ adds a subset $A_G \subseteq \omega_1$ having finite intersection with every element of $([\omega_1]^{\omega^\omega})^V$. Let $B_\alpha := e_\alpha[A_G]$. Then $B_\alpha$ also has finite intersection with every element of $([\omega_1]^{\omega^\omega})^V$. Now for any $\alpha, \alpha' < \omega_2$ distinct,

$$B_\alpha \cap B_{\alpha'} \subseteq (A_\alpha \cap A_{\alpha'}) \cap B_\alpha$$

is finite. $\square$

Now we consider the question of whether $\mathbb{P}_{\mathfrak{mat}_\omega}$ adds an uncountable subset of $\omega_1$ whose intersection with every member of $[\omega_1]^{\omega^2} \cap V$ is finite (and thus has the crucial property of the generic subset of $\omega_1$ added by $\mathbb{P}_{\mathfrak{mat}_{\omega^2}}$). We show that this doesn't happen, so $\mathbb{P}_{\mathfrak{mat}_\omega}$ satisfies a very weak form of distributivity. A main tool will be the following lemma, which reappears in Section 5.

**Lemma 3.5.** *If* $\{f_\xi : \xi \in \omega_1\} \subseteq {}^\omega\omega$*, then for some* $g \in {}^\omega\omega$*,*

$$|\{\xi \in \omega_1 : \forall n \in \omega(f_\xi(n) < g(n))\}| \geq \omega.$$

*Proof.* Fix $\{f_\xi : \xi \in \omega_1\} \subseteq {}^\omega\omega$. First, find $n_0 \in \omega$ such that $A_0 = \{\xi \in \omega_1 : f_\xi(0) < n_0\}$ is uncountable. Then find $n_1 \in \omega$ such that $A_1 = \{\xi \in A_0 : f_\xi(1) < n_1\}$ is uncountable. Proceeding in this fashion, we have $\langle n_i : i \in \omega\rangle$ and $A_0 \supseteq A_1 \supseteq \ldots$. Let $A = \{\alpha_n : n \in \omega\} \subseteq \omega_1$ be such that $\alpha_n \in A_n$. By construction, if $g(k) = n_k + \sum_{i \in k} f_{\alpha_i}(k)$ for every $k \in \omega$, then $\sup\{f_\alpha(k) : \alpha \in A\} < g(k)$, as desired. □

Actually, we will use the following strengthened version of Lemma 3.5.

**Corollary 3.6.** *Suppose* $k < \omega$ *and there are* $k$ *families* $\{f^i_\xi : \xi \in \omega_1\} \subseteq {}^\omega\omega$, $i < k$. *Then for some* $h \in {}^\omega\omega$*,*

$$\mathrm{ot}(\{\xi \in \omega_1 : \forall i < k \forall n \in \omega(f^i_\xi(n) < h(n))\}) \geq \omega^2.$$

*Proof.* By letting $f_\xi := \sup_{i<k} f^i_\xi$, it suffices to prove the corollary for a single family. Apply Lemma 3.5 $\omega_1$ times to get subsets $A_\zeta \in [\omega_1]^\omega$ and functions $g_\zeta \in {}^\omega\omega$ for $\zeta < \omega_1$ so that:

- for all $\xi \in A_\zeta$, $f_\xi < g_\zeta$ (pointwise).
- for $\zeta < \zeta' < \omega_1$, $\sup(A_\zeta) < \min(A_{\zeta'})$.

*Remark* 3.7. It is possible to replace $\omega^2$ in the Corollary above by any countable ordinal, using the technique of [10] in a straightforward way.

Then applying Lemma 3.5 again, there is $B \in [\omega_1]^\omega$ and $h \in {}^\omega\omega$ so that $g_\zeta < h$ for all $\zeta \in B$. This choice of $h$ works, bounding all $f_\xi$ with $\xi \in \bigcup_{\zeta \in B} A_\zeta$. □

**Theorem 3.8.** *Forcing with* $\mathbb{P} := \mathbb{P}_{\mathfrak{mat}_\omega}$ *does not add an uncountable subset of* $\omega_1$ *whose intersection with every member of* $[\omega_1]^{\omega^2} \cap V$ *is finite.*

*Proof.* Let $\dot{A}$ be a $\mathbb{P}$-name for an uncountable subset of $\omega_1$ and let $p \in \mathbb{P}$ be arbitrary. Find an increasing sequence $\langle \beta_\xi : \xi < \omega_1\rangle$ and $p_\xi = (s_\xi, c_\xi) \leq p$ so that $p_\xi \Vdash \beta_\xi \in \dot{A}$. Then we may thin out and re-index to assume:

1. $\{s_\xi : \xi < \omega_1\}$ forms a head-tail-tail $\Delta$-system (defined in the proof of Theorem 3.2) with root $r$.
2. There is a $k < \omega$ so that for all $\xi < \omega_1$, $\mathrm{ot}(c_\xi) = \omega \cdot k$.
3. $\{\mathrm{Lim}(c_\xi) : \xi < \omega_1\}$ forms a head-tail-tail $\Delta$-system with root $r'$.
4. For each $\xi < \omega_1$, let $\mathrm{tail}(\xi) = \{\alpha \in c_\xi : \min(\mathrm{Lim}(c_\xi) \setminus \alpha) \notin r'\}$. Then $\{\mathrm{tail}(\xi) : \xi < \omega_1\}$ forms a $\Delta$-system with root $r_{\mathrm{tail}}$.

To get condition 4, we use the following combinatorial lemma:

**Lemma 3.9.** *Suppose* $\{d_\xi : \xi < \omega_1\}$ *is a set of subsets of* $\omega_1$ *of order-type* $\omega$*, and* $\sup(d_\xi)$ *is strictly increasing in* $\xi$*. Then there is a subset* $X \subseteq \omega_1$ *such that* $|X| = \omega_1$ *and* $\{d_\xi : \xi \in X\}$ *is a head-tail-tail* $\Delta$*-system with a finite root.*

*Proof of Lemma 3.9.* For each $\xi < \omega$ there is $k_\xi < \omega$ so that $|d_{\xi+1} \cap \sup(d_\xi)| = k_\xi$. Fix the value $k$ for $k_\xi$ on an uncountable set $X'$. For each $\xi \in X'$, let $\mathrm{start}(\xi+1)$ be the least $k$ elements of $d_{\xi+1}$. Then there is an uncountable $X'' \subseteq X'$ so that $\{\mathrm{start}(\xi+1) : \xi \in X''\}$ forms a head-tail-tail $\Delta$-system with root $r$. Now thin to get an uncountable $X$ so that for any $\xi < \xi'$ both in $X$, $\sup d_\xi < \min(\mathrm{start}(\xi') \setminus r)$.

We can now check that for any $\xi < \xi'$ both in $X$,

$$d_\xi \cap d_{\xi'} \subseteq \sup(d_\xi) \cap d_{\xi'} \subseteq \sup(d_\xi) \cap \mathrm{start}(\xi') = r,$$

and the least element of $d_{\xi'} \setminus r$ is above $\sup(d_\xi)$. □

Now condition 4 can be achieved by breaking each $c_\xi$ into $k$ blocks of order-type $\omega$ and applying Lemma 3.9 to the families of corresponding blocks to build a head-tail-tail $\Delta$-system.

For each $\xi < \omega_1$, let

$$\mathrm{head}(\xi) = \{\alpha \in c_\xi : \min(\mathrm{Lim}(c_\xi) \setminus \alpha) \in r'\}.$$

We will find $x \in [\omega_1]^{\omega^2}$ and $y \in [\omega_1]^{<\omega^2}$ so that for each $\xi \in x$, $\mathrm{head}(\xi) \subseteq y$.

For each $\alpha \in r'$, find $\langle \alpha_n : n < \omega \rangle$ increasing and cofinal in $\alpha$. Let $\varphi_n^\alpha : [\alpha_n, \alpha_{n+1}) \to \omega$ be injective. Associate to each $\xi < \omega_1$ and each $\alpha \in r'$ the function $f_\xi^\alpha$ defined by

$$f_\xi^\alpha(n) := \sup \varphi_n^\alpha[c_\xi \cap [\alpha_n, \alpha_{n+1})].$$

By Corollary 3.6, there are $B \in [\omega_1]^{\omega^2}$ and $h : \omega \to \omega$ so that $f_\xi^\alpha < h$ pointwise for all $\alpha \in r'$ and all $\xi \in B$. Define

$$x := \{\beta_\xi : \xi \in B\}$$

$$y := \bigcup_{\alpha \in r', n < \omega} \varphi_n^\alpha[h(n)]$$

and

$$p^* := (r, r_{\mathrm{tail}} \cup y).$$

Note that $\mathrm{ot}(y) \leq \omega k$.

To finish the proof, we claim that $p^*$ forces $x \cap \dot{A}$ is infinite.

To show this, let $(s, c) \leq p^*$ be arbitrary and $x_0 \subseteq x$ be finite, and we will find a strengthening of $(s, c)$ which forces a member of $x \setminus x_0$ into $\dot{A}$.

The set $C_0 := \{\xi < \omega_1 : (s_\xi \setminus r) \cap c \neq \emptyset\}$ has order-type $< \omega^2$ since $\{s_\xi : \xi < \omega_1\}$ forms a head-tail-tail $\Delta$-system with root $r$.

Now consider the set $C_1 := \{\xi \in B : s \cap (c_\xi \setminus (r_{\mathrm{tail}} \cup y)) \neq \emptyset\}$. By the construction of $B$ and $p^*$, if $\xi \in B$ then $c_\xi \setminus y \subseteq \mathrm{tail}(\xi)$. But $\{\mathrm{tail}(\xi) : \xi < \omega_1\}$ forms a $\Delta$-system with root $r_{\mathrm{tail}}$. So

$$|\{\xi \in B : s \cap (c_\xi \setminus (r_{\mathrm{tail}} \cup y)) \neq \emptyset\}| < \omega.$$

Putting this all together, there exists $\xi \in B \setminus (C_0 \cup C_1)$ with $\beta_\xi \in x \setminus x_0$. Now $(s, c)$ is compatible with $p_\xi$ and a common strengthening forces $\beta_\xi \in \dot{A}$. □

Theorem 3.8 does not extend to iterations.

**Proposition 3.10.** *$\mathbb{P}_{\mathfrak{mat}_\omega} * \dot{\mathbb{P}}_{\mathfrak{mat}_\omega}$ adds an uncountable subset of $\omega_1$ whose intersection with every member of $[\omega_1]^{\omega^2} \cap V$ is finite.*

*Proof.* Let $G_0 * G_1$ be generic for the iteration, and let $A_0 = \bigcup_{p \in G_0} s_p$ and $A_1 = \bigcup_{p \in G_1} s_p$ be the subsets of $\omega_1$ they define in the generic extension. By a straightforward density argument, $A_1$ has infinite intersection with every member of $[\omega_1]^{\omega^2} \cap V[G_0]$, so $A_0 \cap A_1$ is uncountable.

Let $y \in [\omega_1]^{\omega^2} \cap V$ be arbitrary. Then $y \cap (A_0 \cap A_1) = (y \cap A_0) \cap A_1$. Since $y \in V$ we have $\mathrm{ot}(y \cap A_0) \leq \omega$ and since $y \cap A_0 \in V[G_0]$ we conclude $\mathrm{ot}((y \cap A_0) \cap A_1) < \omega$. □

## 4. Cohen forcing

In this section, let $\mathbb{C}$ be Cohen forcing and for a cardinal $\kappa$ let $\mathbb{C}_\kappa$ be the forcing adding $\kappa$ Cohen reals with finite support. For concreteness, let $\mathbb{C}_\kappa$ be the set of all partial functions $\kappa \to 2$ with finite domain, ordered by reverse inclusion.

**Proposition 4.1.** *Forcing with $\mathbb{C}$ does not change the value of $\overset{\bullet}{|}$.*

*Proof.* Let $G$ be generic for $\mathbb{C}$ over $V$. As Cohen forcing is $(\omega_1, \omega_1)$-semidistributive, a $\overset{\bullet}{|}$-sequence in the ground model remains a $\overset{\bullet}{|}$-sequence in the extension, so $\overset{\bullet}{|}^{V[G]} \leq \overset{\bullet}{|}^{V}$. On the other hand, we can construct a $\overset{\bullet}{|}$-sequence in the ground model from a name for a $\overset{\bullet}{|}$-sequence in the extension. More generally, we can prove:

**Lemma 4.2.** *Suppose $\mathbb{P}$ is a forcing poset preserving $\omega_1$. Then for any generic $G$, $\overset{\bullet}{|}^{V} \leq \max\{\overset{\bullet}{|}^{V[G]}, d(\mathbb{P})\}$. Here $d(\mathbb{P})$ refers to the density of $\mathbb{P}$, that is the minimal cardinality of a dense subset of $\mathbb{P}$.*

Working in $V$, let $\langle \dot{d}_\xi : \xi < \overset{\bullet}{|}^{V[G]} \rangle$ be a name for a $\overset{\bullet}{|}$-sequence of minimal cardinality in $V^{\mathbb{P}}$ and let $\langle p_i : i < d(\mathbb{P}) \rangle$ be an enumeration of a suitable dense $D \subseteq \mathbb{P}$. Let $\mathcal{A}_i^\xi = \{A \in [\omega_1]^{\omega_1} : p_i \Vdash \dot{d}_\xi \subseteq A\}$. Then we claim that the set

$$\left\{ \bigcap \mathcal{A}_i^\xi : \xi < \overset{\bullet}{|}^{V[G]}, i < d(\mathbb{P}) \right\}$$

is a $\overset{\bullet}{|}$-sequence in $V$. This set has cardinality $\max\{\overset{\bullet}{|}^{V[G]}, d(\mathbb{P})\}$ and each member is infinite (since each $\dot{d}_\xi$ is forced to be infinite). Finally, for each $A \in [\omega_1]^{\omega_1}$ there are $i, \xi$ such that $p_i \Vdash \dot{d}_\xi \subseteq A$, so $\bigcap \mathcal{A}_i^\xi \subseteq A$. $\square$

The $(\omega_1, \omega_1)$-semidistributivity together with other simple properties also easily implies that $\mathbb{C}$ has no effect on $\mathfrak{cc}(\omega_1, \omega)$ or $\mathfrak{mat}_\omega([\omega_1]^{\omega_1}, \omega)$.

It is not difficult to show that $\mathbb{C}_{\omega_1}$ adds an uncountable subset of $\omega_1$ with no infinite subset in the ground model (i.e., $\mathbb{C}_{\omega_1}$ is not $(\omega, \omega_1)$-semidistributive). Therefore, for $\kappa$ regular and uncountable, $\mathbb{C}_\kappa$ makes the value of $\overset{\bullet}{|}$ at least $\kappa$. Alternatively, it is not difficult to see that $\mathsf{MA}(\mathbb{C}_{\omega_1})$ implies $\overset{\bullet}{|} = 2^\omega$.

We also observe that after forcing with $\mathbb{C}_\kappa$ for any $\kappa$, $([\omega_1]^\omega)^V$ is countably matching for $[\omega_1]^{\omega_1}$. In fact, since $\mathbb{C}_\kappa$ does not add dominating reals, Proposition 2.9 implies that $V$ is $([\omega_1]^\omega, \mathrm{ot}(\omega))$-matching in $V[G]$, so $([\omega_1]^\omega)^V$ is even countably matching for $[\omega_1]^\omega$.

We will now show that $\mathfrak{cc}(\omega_1, \omega)$ is not affected by adding $\omega_2$ Cohen reals. This is not obvious from Fact 1.1 since as noted $\overset{\bullet}{|} > \aleph_1$ in this model.

Say that a partial order $\mathbb{P}$ is $< \kappa$-centered if and only if for some $\lambda < \kappa$ we can write $\mathbb{P} = \bigcup_{\alpha \in \lambda} \mathbb{P}_\alpha$ where $\mathbb{P}_\alpha \subseteq \mathbb{P}$ is centered, that is so that every finite collection of conditions in $\mathbb{P}_\alpha$ has a lower bound in $\mathbb{P}_\alpha$. First, we present a well-known but somewhat surprising fact:

**Fact 4.3.** Let $\log_2(\aleph_2)$ be the least cardinal $\kappa$ such that $2^\kappa \geq \omega_2$. Then $\mathbb{C}_{\omega_2}$ is $< \log_2(\aleph_2)^+$-centered. Therefore, $\mathbb{C}_{\omega_2}$ is $\omega_1$-centered, and if $\mathsf{CH}$ fails then $\mathbb{C}_{\omega_2}$ is $\sigma$-centered.

*Proof.* If $\mathsf{CH}$ fails then $\log_2(\aleph_2) = \aleph_0$, and otherwise $\log_2(\aleph_2) = \aleph_1$. It is well-known that there is an independent family on $\log_2(\aleph_2)$ of size $\aleph_2$, i.e., a family

$\langle f_\alpha : \alpha < \omega_2 \rangle$ of functions $\log_2(\aleph_2) \to 2$ such that for any finite subset $A \subseteq \omega_2 \times 2$, there is $\xi < \log_2(\aleph_2)$ so that for each $(\alpha, \epsilon) \in A$, $f_\alpha(\xi) = \epsilon$.

Now for $\xi < \log_2(\aleph_2)$, let $D(\xi)$ be the set of all conditions $p \in \mathbb{C}_{\omega_2}$ so that for all $\alpha \in \operatorname{supp}(p)$, $p(\alpha) = f_\alpha(\xi)$. Clearly, $D(\xi)$ is centered. Since the functions $f_\alpha$ are independent, any $p \in \mathbb{C}_{\omega_2}$ is in $D(\xi)$ for some $\xi < \log_2(\aleph_2)$. This proves the fact. $\square$

**Theorem 4.4.** *Let $\mathfrak{cc}(\omega_1, \omega) = \kappa$. After forcing with $\mathbb{C}_{\omega_2}$, the value of $\mathfrak{cc}(\omega_1, \omega)$ remains $\kappa$.*

*Proof.* First we consider the case where $\mathsf{CH}$ fails. Then Fact 4.3 implies that $\mathbb{C}_{\omega_2}$ is $\sigma$-centered, so let $D(n)$, $n < \omega$, be centered subsets whose union is $\mathbb{C}_{\omega_2}$.

Suppose that there is in $V^{\mathbb{C}_{\omega_2}}$ a name for an antichain $\langle \dot{A}_\alpha : \alpha < \kappa \rangle$.

For each $\alpha < \kappa$, let $A'_\alpha \subseteq \omega_1$ be cofinal in $\omega_1$ so that:

(1) for every $\xi \in A'_\alpha$, there is $p^\alpha_\xi \Vdash \xi \in \dot{A}_\alpha$.
(2) there is $n_\alpha < \omega$ so that $p^\alpha_\xi \in D(n_\alpha)$ for all $\xi \in A'_\alpha$.
(3) $\langle \operatorname{dom}(p^\alpha_\xi) : \xi \in A'_\alpha \rangle$ forms a $\Delta$-system with root $r^\alpha$, and there is $p^\alpha$ so that $p^\alpha_\xi \restriction r^\alpha = p^\alpha$.

We obtain $A'_\alpha$ by using the fact that $\dot{A}_\alpha$ is forced to be cofinal in $\omega_1$, thinning to stabilize the centered piece containing the $p^\alpha_\xi$, and then using the $\Delta$-system lemma.

Since $\kappa$ is regular, there are $n < \omega$ and $X \subseteq \kappa$ with $|X| = \kappa$ such that $n_\alpha = n$ for all $\alpha \in X$. Take $\alpha_0, \alpha_1$ distinct in $X$ such that $|A'_{\alpha_0} \cap A'_{\alpha_1}| \geq \omega$. Let $p^* := p^{\alpha_0} \cup p^{\alpha_1}$. We will show that $p^*$ forces that $\dot{A}_{\alpha_0} \cap \dot{A}_{\alpha_1}$ is infinite, a contradiction.

For this, fix $B \subseteq A'_{\alpha_0} \cap A'_{\alpha_1}$ of order-type $\omega$. We will show that for every $\xi' < \sup(B)$, the set of conditions $p \leq p^*$ forcing some $\xi \in B \setminus \xi'$ into $\dot{A}_{\alpha_0} \cap \dot{A}_{\alpha_1}$ is dense. So let $q \leq p^*$. Since $\operatorname{dom}(q)$ is finite, and $\langle \operatorname{dom}(p^\alpha_\xi) : \xi \in A'_\alpha \rangle$ forms a $\Delta$-system for $\alpha = \alpha_0, \alpha_1$, there is $\xi \in B \setminus \xi'$ so that $\operatorname{dom}(q) \cap \operatorname{dom}(p^{\alpha_0}_\xi) \subseteq r^{\alpha_0}$ and $\operatorname{dom}(q) \cap \operatorname{dom}(p^{\alpha_1}_\xi) \subseteq r^{\alpha_1}$. Then $p^{\alpha_0}_\xi$, $p^{\alpha_1}_\xi$, and $q$ are all compatible, and their union forces $\xi$ into $\dot{A}_{\alpha_0} \cap \dot{A}_{\alpha_1}$. This finishes the proof in the first case.

If $\mathsf{CH}$ holds, this result is proved in Section 5 of [3]. We remark that the argument above adapts for this case as well, using the $< \omega_2$-centeredness of $\mathbb{C}_{\omega_2}$ (Fact 4.3) and the $\mathsf{CH}$ to ensure that conditions forcing corresponding ordinals into the members of the antichain come from the same centered piece. $\square$

## 5. Hechler forcing

Proposition 2.9 gives a strong limitation about what happens to the matching numbers under forcings which do not add dominating reals, so our focus shifts to forcing which do add them, such as Hechler forcing.

Furthermore, we are interested in whether $\mathfrak{b}$ can be large while the values of the cardinal invariants remain small. For example, it is open whether $⫯ = \aleph_1$ implies $\mathfrak{b} = \aleph_1$. A natural way to increase $\mathfrak{b}$ is to iterate forcings which add dominating reals.

Let $\mathbb{D}$ be the Hechler poset, which is the set of pairs $(s, f)$ such that $s \in {}^{<\omega}\omega$, $f \in \omega^\omega$, and $s \subseteq f$, with ordering $(s, f) \leq (t, g)$ iff $s \supseteq t$ and $f(n) \geq g(n)$ for all $n$. We call $s$ the *stem* of the condition.

**Proposition 5.1.** *$\mathbb{D}$ is $(\omega_1, \omega_1)$-semidistributive if and only if $\mathfrak{b} > \omega_1$.*

*Proof.* First suppose $\mathfrak{b} > \omega_1$. Let $\dot{A}$ be a $\mathbb{D}$-name for an uncountable subset of $\omega_1$. Then find an increasing sequence $\langle \beta_\xi : \xi < \omega_1 \rangle$ in $\omega_1$ and conditions $p_\xi = (s_\xi, f_\xi)$, $\xi < \omega_1$, so that $p_\xi \Vdash \beta_\xi \in \dot{A}$. By thinning out, we may assume that all conditions have the same stem $s$. Since $\mathfrak{b} > \omega_1$, $\{f_\xi : \xi < \omega_1\}$ is $<^*$-bounded, so find a bound $g$. By thinning out again, we may assume that there is some $n < \omega$ so that $\{f_\xi(i) : f_\xi(i) > g(i)\} \subseteq n$ for all $\xi < \omega_1$. Then define $h := \max(n, g)$ on $\omega \setminus \mathrm{dom}(s)$ (and $h = s$ on $\mathrm{dom}(s)$). The condition $(s, h)$ is a common lower bound for $\{p_\xi : \xi < \omega_1\}$ and therefore $(s, h) \Vdash \{\beta_\xi : \xi < \omega_1\} \subseteq \dot{A}$.

Now suppose $\mathfrak{b} = \omega_1$. Fix a $<^*$-increasing, unbounded sequence of functions $B = \langle g_\xi : \xi < \omega_1 \rangle$ and a bijection $\varphi : {}^{<\omega}\omega \times B \to \omega_1$. Define a $\mathbb{D}$-name $\dot{A}$ for a subset of $\omega_1$ so that $(s, f) \Vdash \alpha \in \dot{A}$ if and only $(s, f) \leq \varphi^{-1}(\alpha)$. Now $\dot{A}$ is forced to be uncountable, since for any $\alpha_0 < \omega_1$ and $(s, f) \in \mathbb{D}$ there is some $\xi$ so that $\varphi(s, g_\xi) > \alpha_0$, and $(s, g_\xi)$ and $(s, f)$ are compatible.

It remains to see that $\dot{A}$ contains no ground model uncountable set. Suppose otherwise, so there are $(s, f) \in \mathbb{D}$ and $A' \in [\omega_1]^{\omega_1} \cap V$ so that $(s, f) \Vdash A' \subseteq \dot{A}$. Let

$$I = \{g_\xi : \exists \alpha \in A' \exists t \in {}^{<\omega}\omega \, (\varphi^{-1}(\alpha) = (t, g_\xi))\}.$$

The set $I$ must be uncountable since $A'$ is uncountable and ${}^{<\omega}\omega$ is countable. Since $B$ is $<^*$-increasing, $I$ is unbounded. But by the definition of the name $\dot{A}$, we have $(s, f) \leq \varphi^{-1}(\alpha)$ for all $\alpha \in A'$, which implies that $f(n) \geq g_\xi(n)$ for all $g_\xi \in I$ and $n \geq \mathrm{length}(s)$, a contradiction. □

*Remark* 5.2. The same argument as in Proposition 5.1 shows more generally that for regular $\kappa$, $\mathbb{D}$ is $(\kappa, \kappa)$-semidistributive if $\mathfrak{b} > \kappa$ and is not $(\kappa, \kappa)$-semidistributive if $\mathfrak{b} = \kappa$. However, because $d(\mathbb{D}) = \mathfrak{d}$ and generally for any $\lambda$ with $\mathrm{cf}(\lambda) > d(\mathbb{P})$ we must have that $\mathbb{P}$ is $(\lambda, \lambda)$-semidistributive by a pigeonhole argument, $\mathbb{D}$ is for example $(\mathfrak{d}^+, \mathfrak{d}^+)$-semidistributive.

The argument in Proposition 5.1 showed that when $\mathfrak{b} > \omega_1$, if $\{f_\xi : \xi \in \omega_1\} \subseteq {}^\omega\omega$ then for some $g \in {}^\omega\omega$,

$$|\{\xi \in \omega_1 : \forall n \in \omega (f_\xi(n) < g(n))\}| = \omega_1.$$

Compare this to Lemma 3.5, which says that (regardless of the value of $\mathfrak{b}$) if $\{f_\xi : \xi \in \omega_1\} \subseteq {}^\omega\omega$, then for some $g \in {}^\omega\omega$,

$$|\{\xi \in \omega_1 : \forall n \in \omega (f_\xi(n) < g(n))\}| \geq \omega.$$

It is then straightforward to see

**Proposition 5.3.** *$\mathbb{D}$ is always $(\omega, \omega_1)$-semidistributive.*

We can apply the results about semidistributivity to prove a partial result on the preservation of $\mathfrak{cc}(\omega_1, \omega)$ by Hechler forcing.

**Theorem 5.4.** *Suppose either* $\mathsf{CH}$ *holds or $\mathfrak{b} > \omega_1$. Let $\mathfrak{cc}(\omega_1, \omega) = \kappa$. After forcing with $\mathbb{D}$, the value of $\mathfrak{cc}(\omega_1, \omega)$ remains $\kappa$.*

*Proof.* If $\mathsf{CH}$ holds in $V$, then it holds in the extension by $\mathbb{D}$ and hence $\mathfrak{cc}(\omega_1, \omega)$ remains $\omega_1$.

If $\mathfrak{b} > \omega_1$, then suppose that there is in $V^{\mathbb{D}}$ a name for an antichain $\langle \dot{A}_\alpha : \alpha < \kappa \rangle$. For each $\alpha < \kappa$, find $A'_\alpha \in [\omega_1]^{\omega_1}$ and $p_\alpha = (s_\alpha, f_\alpha)$ so that $p_\alpha \Vdash A'_\alpha \subseteq A_\alpha$. By thinning, we may assume that there is $s \in {}^{<\omega}\omega$ such that $s_\alpha = s$ for all $\alpha < \kappa$.

Since $\mathfrak{cc}(\omega_1, \omega) = \kappa$ in the ground model, there are $\alpha < \beta < \kappa$ so that $A'_\alpha \cap A'_\beta$ is infinite. But then

$$(s, \max(f_\alpha, f_\beta)) \Vdash A'_\alpha \cap A'_\beta \subseteq \dot{A}_\alpha \cap \dot{A}_\beta,$$

contradicting that $\langle \dot{A}_\alpha : \alpha < \kappa \rangle$ was forced to be an antichain. □

Unfortunately, the hypothesis $\mathfrak{b} > \omega_1$ does not persist through iterations of the forcing (even the two-step iteration, as shown by [6]) and we do not know if the iteration preserves the value of $\mathfrak{cc}(\omega_1, \omega)$, or whether it is consistent that $\mathfrak{b} > \mathfrak{cc}(\omega_1, \omega)^-$.

## 6. Almost disjoint $\stick$ sequences

In this section, we consider the problem of the existence of $\stick$ sequences which are themselves almost disjoint families.

For a cardinal $\kappa$, define $\stick^{\mathrm{ad}}(\kappa)$ to be the minimal cardinality of an almost disjoint subset $X \subseteq [\kappa^+]^\kappa$ such that for every $y \in [\kappa^+]^{\kappa^+}$ there exists $x \in X$ with $x \subseteq y$. Here $X$ is said to be almost disjoint if $|x \cap x'| < \kappa$ for every $x, x' \in X$. We denote $\stick^{\mathrm{ad}}(\omega)$ by $\stick^{\mathrm{ad}}$. In [8] it was observed that $\stick = \stick^{\mathrm{ad}}$ so long as $\stick < 2^\omega$. In [7] it was observed that if $\stick = \omega_1$, then $\stick = \stick^{\mathrm{ad}}$. Using a result of Balcar and Vojtáš [1] concerning the almost disjoint refinability of positive sets for certain tall ideals over $\omega$, we may show that $\stick = \stick^{\mathrm{ad}}$ outright.

**Theorem 6.1.** *$\stick = \stick^{ad}$.*

*Proof.* For $\stick < 2^\omega$ we have already seen that $\stick = \stick^{\mathrm{ad}}$, so it suffices to show that $[\omega_1]^{\omega^2}$ can be almost disjointly refined. Fix $\delta \in \lim(\omega_1)$ and let $\langle \alpha^\delta_n : n \in \omega \rangle \subseteq \delta$ be increasing and cofinal with $\alpha^\delta_0 = 0$ and $|[\alpha^\delta_n, \alpha^\delta_{n+1})| \geq n$ for every $n \in \omega$. Let $Q = \{q_n : n \in \omega\}$ with $q_n = [\alpha^\delta_n, \alpha^\delta_{n+1})$ be the corresponding partition of $\delta$ so that in particular $Q$ comprises pieces which are not eventually bounded by any finite cardinality. Let $\mathcal{Y}^+(Q)$ denote the collection of positive sets for the ideal over $\delta$ generated by $Q$ along with the subsets of $\delta$ whose intersections with the $q_n$'s are eventually bounded in finite cardinality. By Theorem A in [1], $\mathcal{Y}^+(Q)$ has an almost disjoint refinement. Let $X_\delta = \{x \in [\delta]^\omega : |x \cap q_n| < \omega \text{ and } \operatorname{limsup}_n |x \cap q_n| = \omega\}$, and note that $X_\delta \subseteq \mathcal{Y}^+(Q)$ and comprises elements of order type $\omega$. So there exits $A_\delta \subseteq [\delta]^\omega$ such that $A_\delta$ is an almost disjoint collection and for every $x \in X_\delta$ there is $a \in A_\delta$ with $a \subseteq x$. Note that $A = \{a \in [\omega_1]^\omega : a \in A_\delta \text{ for some } \delta \in \lim(\omega_1)\}$ is an almost disjoint collection, and if $x \in [\omega_1]^{\omega^2}$ there exists $y \in X_{\sup(x)}$ with $y \subseteq x$, so then for some $a \in A_{\sup(x)} \subseteq A$, $a \subseteq y \subseteq x$. That is, $A$ is an almost disjoint refinement of $[\omega_1]^{\omega^2}$. □

Let $\mathrm{MAD}(\kappa)$ denote the spectrum of cardinalities of maximal almost disjoint collections of $\kappa$-sized subsets of $\kappa$ modulo $< \kappa$. In [8] it is observed that any $\lambda$-sized collection of elements in $[\kappa]^\kappa$ of cardinality less than $\mathfrak{cc}(\kappa, \kappa)^-$ can be almost disjointly refined. Balcar and Vojtáš's Theorem A may be generalized in certain circumstances [8]:

**Proposition 6.2.** *Let $\kappa > \omega$ be regular with the additional property that $MAD(\kappa) \cap (\kappa, 2^\kappa] = \{2^\kappa\}$. Let $\{q_\beta : \beta \in \kappa\} \subseteq P(\kappa)$ be a partition of $\kappa$ such that $|\{\beta \in \kappa : |q_\beta| \geq \omega\}| = \kappa$. Then $\{x \in [\kappa]^\kappa : |\{\beta \in \kappa : |x \cap q_\beta| \geq \omega\}| = \kappa\}$ can be almost disjointly refined.*

Note that the combinatorial hypotheses of Proposition 6.2 hold if $\kappa$ is regular and $2^\kappa = \kappa^+$. Using 6.2, we can generalize Theorem 6.1 to larger $\kappa$ in certain circumstances.

**Proposition 6.3.** *If $\kappa > \omega$ is regular with the additional property that $MAD(\kappa) \cap (\kappa, 2^\kappa] = \{2^\kappa\}$, $⫯_\kappa = ⫯^{ad}_\kappa$.*

*Proof.* The proof is analogous to the argument in Theorem 6.1, this time referencing Proposition 6.2 to show that $[\kappa^+]^{\kappa^2}$ can be almost disjointly refined by considering a suitable $\kappa$-ladder to every $\alpha \in \kappa^+$ which can accommodate a cofinal $\mathrm{ot}(\kappa^2)$-sequence, and so on. □

## 7. Open questions

Many open questions remain. The first question addresses matching numbers for countable subsets of $\omega_1$.

*Question* 7.1. Is it consistent that $\mathfrak{mat}_\delta([\omega_1]^{\delta\cdot\omega^2}, \omega) < \mathfrak{mat}_\delta([\omega_1]^{\delta\cdot\omega}, \omega)$?

We would like to have a more complete picture of the matching numbers for uncountable subsets of $\omega_1$.

*Question* 7.2. What are the equalities between quantities of the form $\mathfrak{mat}_\delta([\omega_1]^{\omega_1}, \omega)$ provable in $\mathsf{ZFC}$?

The general question of computing $\mathfrak{cc}(\omega_1, \omega)$ in different forcing extensions is interesting. In particular, we ask:

*Question* 7.3. Can $\mathbb{D}$ change the value of $\mathfrak{cc}(\omega_1, \omega)$? What about Hechler iterations (finite support, or even the mixed-support iterations considered in Brendle [4])?

By the results of Section 5, the question for a single Hechler forcing is only of interest when $\mathfrak{b} = \omega_1$ and $\neg\mathsf{CH}$.

We saw that Lemma 3.5 is used both in understanding the semidistributivity of Hechler forcing as well as in the proof to Theorem 3.8. It is natural to consider higher analogues, like the following question about $\omega_1$-Baire space:

*Question* 7.4. If $\{f_\xi : \xi \in \omega_2\} \subseteq {}^{\omega_1}\omega_1$, when is it the case that there exists a $g \in {}^{\omega_1}\omega_1$ with $|\{\xi \in \omega_1 : \forall \alpha \in \omega_1 (f_\xi(\alpha) < g(\alpha))\}| \geq \omega_1$?

*Remark* 7.5. The statement that there always exists such a function $g$ in Question 7.4 is consistently true modulo large cardinals—in the model obtained by Lévy collapsing a measurable cardinal $\lambda$ to $\omega_2$, the resulting strong ideal over $\omega_2$ can be used to mimic the proof of Lemma 3.5.

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