ON THE RELATIONSHIP BETWEEN MUTUAL AND TIGHT STATIONARITY

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Abstract. We construct a model where every increasing ω-sequence of regular cardinals carries a mutually stationary sequence which is not tightly stationary, and show that this property is preserved under a class of Prikry-type forcings. Along the way, we give examples in the Cohen and Prikry models of ω-sequences of regular cardinals for which there is a non-tightly stationary sequence of stationary subsets consisting of cofinality ω_1 ordinals, and show that such stationary sequences are mutually stationary in the presence of interleaved supercompact cardinals.

1. Introduction

Foreman and Magidor [10] introduced two new concepts of stationarity: *mutual stationarity* and *tight stationarity*. Each of these notions is a property of sequences $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ where $S_\xi \subseteq \kappa_\xi$ and $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ is a sequence of regular cardinals cofinal in some singular cardinal κ. A more basic concept is that of a *stationary sequence*, which is simply a sequence $\vec{S}$ where $S_\xi \subseteq \kappa_\xi$ is stationary for all but boundedly many $\xi < \mathrm{cf}(\kappa)$. We will focus on the case where $\mathrm{cf}(\kappa) = \omega$, and where each S_ξ consists of ordinals of some fixed uncountable cofinality η (existence of a mutually stationary sequence which does not meet this second condition implies instances of Chang's conjecture, see [9]). An immediate relationship between these concepts is:

$$\text{tightly stationary} \quad\Longrightarrow\quad \text{mutually stationary} \quad\Longrightarrow\quad \text{stationary}.$$

This paper is motivated by the following question:

Question 1.1. Can there be a sequence of cardinals $\langle \kappa_i : i < \omega\rangle$ so that every mutually stationary sequence on $\Pi_{i<\omega}\kappa_i$ is tightly stationary?

The question was originally asked in [4] for the sequence $\langle \omega_n : n < \omega\rangle$, and Cummings, Foreman, and Magidor [5] produced models where $\langle \omega_n : n < \omega\rangle$ have mutually stationary sequences which are not tightly stationary.

The main result of this paper is a forcing construction which gives a negative answer to Question 1.1, so that there are mutually stationary but not tightly stationary sequences on *every* increasing ω-sequence of regular cardinals. Moreover, this property is absolute to a wide class of forcing extensions, a class which includes the natural candidates for forcing that every mutually stationary sequence is tightly

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stationary on some sequence of regular cardinals. This shows that a positive answer to Question 1.1, if consistently possible, would be very difficult to achieve, and impossible with many of the standard tools.

The proof of the main result involves separately ensuring that the following two properties hold in the final model:

(1) Every ω-sequence of regular cardinals carries a stationary sequence which is not tightly stationary.
(2) Every stationary sequence is mutually stationary, provided the regular cardinals which carry the sequence are spaced sufficiently far apart.

Whether property (1) is just a theorem of ZFC is an interesting question that, to the authors' knowledge, remains open. There is some partial progress in Section 3, where we give some conditions under which there is a stationary but not tightly stationary sequence, and use forcing to build a model where every sequence of regular cardinals has such a sequence. In Section 4 we show that there is always a stationary not tightly stationary sequence on many sequences of regular cardinals associated with Prikry forcing. Along the way, we will develop a general framework for understanding the scales in Prikry-type forcing extensions.

Statements along the lines of property (2) have been the focus of much attention in previous work in this area. One of the main results of [10] was that every stationary sequence of subsets concentrating on ordinals of countable cofinality is mutually stationary. For uncountable cofinality, they showed that in L there is a stationary sequence of subsets of ordinals of cofinality ω_1 which is not mutually stationary. Cummings, Foreman, and Magidor [5] showed that in the Prikry extension, every stationary sequence is mutually stationary on the product of the generic sequence. Then Koepke [15] adapted this argument to force so that every sequence of stationary sets on $\langle \aleph_{2n+1} : n < \omega \rangle$ concentrating on ordinals of cofinality ω_1 is mutually stationary (this construction also uses one measurable cardinal).

In Section 5, we force over the model of Section 3 so that every increasing ω-sequence of regular cardinals has a mutually stationary but not tightly stationary sequence. The construction uses a proper class of supercompact cardinals, and descends from the results of [5] and [15] mentioned above. In section 6, we show that further Prikry extensions from the model of Section 5 cannot add increasing ω-sequences of regular cardinals where every mutually stationary sequence is tightly stationary.

2. Preliminaries

Let κ be a singular cardinal, and $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ a sequence of regular cardinals cofinal in κ.Take $\theta = (2^{2^\kappa})^+$ and let $\mathcal{A}$ be an algebra on $H(\theta)$, i.e., a structure on $H(\theta)$ with countably many functions in the language. If $M \prec \mathcal{A}$ is an elementary substructure, then define the *characteristic function of* M as $\chi_M : \xi \mapsto \sup(M \cap \kappa_\xi)$. We say M is *tight* if $M \cap \prod_{\xi < \mathrm{cf}(\kappa)} \kappa_\xi$ is cofinal in $\prod (M \cap \kappa_\xi)$.

Suppose $S_\xi \subseteq \kappa_\xi$ for all $\xi < \mathrm{cf}(\kappa)$. The sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ is *mutually stationary* if for any algebra $\mathcal{A}$ on $H(\theta)$ there is $M \prec \mathcal{A}$ such that $\{\xi : \chi_M(\xi) \notin S_\xi\}$ is bounded in $\mathrm{cf}(\kappa)$ (we say that χ_M *meets* $\vec{S}$). The sequence $\vec{S}$ is *tightly stationary* if for every $\mathcal{A}$ on $H(\theta)$, a tight structure $M \prec \mathcal{A}$ as in the previous definition can be chosen.

A *scale* is an increasing unbounded sequence $\langle f_\alpha : \alpha < \lambda\rangle$ in $(\prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi, <^*)$, where $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ are regular cardinals cofinal in κ and $f <^* g$ if and only if $\{\xi : f(\xi) \geq g(\xi)\}$ is bounded in $\mathrm{cf}(\kappa)$. Scales were previously considered in the context of mutual and tight stationarity in [5] and [6]. A basic result of pcf theory due to Shelah [18] says that for singular κ, there is some sequence of regular cardinals which carries a scale of length κ^+. A scale is *continuous* if for every $\beta < \lambda$ of cofinality $> \mathrm{cf}(\kappa)$, if there is an exact upper bound for $\langle f_\alpha : \alpha < \beta\rangle$ (i.e., a $<^*$-upper bound g such that $\langle f_\alpha : \alpha < \beta\rangle$ is cofinal in $\prod_\xi g(\xi)$) then f_β is such a bound.

In [1], it was shown how the scale relates sequences on the $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ to subsets of λ.

Definition 2.1. Suppose $S_\xi \subseteq \kappa_\xi$ for each $\xi < \mathrm{cf}(\kappa)$. Then define

$$\mu(\vec{S}) = \{\alpha : f_\alpha \text{ meets } \vec{S}\}.$$

Let $S'_\xi = \kappa_\xi \setminus S_\xi$. Then define $\nu(\vec{S}) = \lambda \setminus \mu(\langle S'_\xi\rangle)$.

The following lemma from [1] relates tight stationarity to the μ function of Definition 2.1.

Lemma 2.2. *Let η be an uncountable regular cardinal in the interval $(\mathrm{cf}(\kappa), \kappa_0)$. Suppose $S_\xi \subseteq \kappa_\xi \cap \mathrm{Cof}(\eta)$. Then $\vec{S}$ is tightly stationary iff $\mu(\vec{S}) \cap \mathrm{Cof}(\eta) \cap \mathrm{Good}$ is stationary in λ, where* Good *is the set of good points.*

3. Stationary but not tightly stationary sequences

We collect some examples of situations for which there are stationary, not tightly stationary sequences concentrating on a fixed uncountable cofinality.

As mentioned in the introduction, previous examples were found in [5] (mutually but not tightly stationary), and in [10],[16] (stationary but not mutually stationary). Another example was found in [1], where it was shown that every sequence of regular cardinals admitting a tree-like scale has a stationary but not tightly stationary sequence. There are not many examples known of sequences of regular cardinals which have no tree-like scales. For one example, see Gitik [11], where such a sequence is identified in the generic extension by a Prikry-type forcing.

An important definition in [1] was that of a careful sequence. Suppose that $\langle \kappa_n : n < \omega\rangle$ carries a scale $\langle f_\alpha : \alpha < \lambda\rangle$. A sequence $\vec{S} = \langle S_n : n < \omega\rangle$ with $S_n \subseteq \kappa_n$ for all $n < \omega$ is *careful* if $\mu(\vec{S}) = \nu(\vec{S})$. Any sequence of regular cardinals with a tree-like scale admits a careful sequence of stationary co-stationary subsets, as do many sequences of regular cardinals associated with Prikry forcing.

Proposition 3.1. *Suppose there is a careful sequence of stationary, co-stationary subsets. Then there is a sequence of stationary sets which is not tightly stationary.*

Proof. Let $\vec{S}$ be a careful sequence of stationary, co-stationary subsets. Take $X \subseteq \omega$ which is infinite and whose complement is also infinite. Define $\vec{T} = \langle T_n : n < \omega\rangle$ by $T_n = S_n$ if $n \in X$ and $T_n = \kappa_n \setminus S_n$ otherwise. Then T_n is a stationary sequence.

Now we check that $\mu(\vec{T}) = \emptyset$. Suppose $\alpha \in \mu(\vec{T})$. Then $f_\alpha(n) \in S_n$ for infinitely many n, so $\alpha \in \nu(\vec{S})$. Similarly, we get $\alpha \in \nu(\langle \kappa_n \setminus S_n\rangle)$, but this is impossible. □

Under favorable cardinal arithmetic, namely a large value of the continuum, we can find stationary sequences which are not tightly stationary on certain sequences of regular cardinals.

Proposition 3.2. *Suppose $\langle \mu_n : n < \omega \rangle$ is an increasing sequence of regular cardinals which carry a scale $\langle f_\alpha : \alpha < \lambda \rangle$, and $2^{\aleph_0} > \lambda$. Then there is a stationary sequence in $\langle \mu_n : n < \omega \rangle$ which is not tightly stationary.*

Proof. Choose a sequence $\langle (S^0_n, S^1_n) : n < \omega \rangle$ where (S^0_n, S^1_n) is a partition of $\mu_n \cap \mathrm{Cof}(\eta)$ into stationary sets for each $n < \omega$. For each $\alpha < \lambda$, let $x_\alpha \in {}^\omega 2$ be defined by setting $x_\alpha(n)$ equal to the unique i so that $f_\alpha(n) \in S^i_n$ if it exists, and 0 otherwise. Let E_0 be the equivalence relation of eventual agreement on ${}^\omega 2$, i.e., $x \equiv y$ iff there is some m so that $x(n) = y(n)$ for all $n \geq m$. The classes of E_0 are countable, so there is some $x \in {}^\omega 2$ so that x is not E_0-equivalent to x_α for any $\alpha < \lambda$.

The sequence $\langle S^{x(n)}_n : n < \omega \rangle$ is not tightly stationary since $\mu(\langle S_n \rangle) = \emptyset$ by the choice of x. □

The basic idea of Proposition 3.2 can be used to force every ω-sequence of regular cardinals to have a stationary sequence which is not tightly stationary.

Suppose $\langle \mu_n : n < \omega \rangle$ is an increasing sequence of regular cardinals cofinal in μ which carries a scale $\langle f_\alpha : \alpha < \lambda \rangle$. If a forcing poset $\mathbb{P}$ has the μ-c.c., it is easy to see that any member of $(\prod_n \mu_n)^{V[G]}$ is $<^*$-below a member of $(\prod_n \mu_n)^V$, so $\langle f_\alpha : \alpha < \lambda \rangle$ remains a scale in $V[G]$. Furthermore, stationary subsets of regular cardinals $> \mu$ are preserved. So a tightly stationary sequence in the ground model must remain tightly stationary in the extension.

Recall that the Cohen forcing $\mathrm{Add}(\omega, \lambda)$ is the poset of partial functions $\lambda \times \omega \to 2$ with finite domain ordered by reverse inclusion.

Proposition 3.3. *Suppose $\langle \mu_n : n < \omega \rangle$ is an increasing sequence of regular cardinals and $2^{\aleph_0} < \mu_0$ and $\langle T_n : n < \omega \rangle$ is a stationary sequence on $\langle \mu_n : n < \omega \rangle$. Then forcing with $\mathrm{Add}(\omega, 1)$ adds a sequence $\langle S_n : n < \omega \rangle$ on $\langle \mu_n : n < \omega \rangle$ so that:*

- *$S_n \subseteq T_n$ is stationary for every n,*
- *$\langle S_n : n < \omega \rangle$ is not tightly stationary.*

Proof. Choose in V a sequence $\langle (S^0_n, S^1_n) : n < \omega \rangle$ where (S^0_n, S^1_n) is a partition of T_n into stationary sets for each $n < \omega$. Let $x : \omega \to 2$ be the real added by $\mathrm{Add}(\omega, 1)$. Then the sequence $\langle S^{x(n)}_n : n < \omega \rangle$ is stationary, since $\mathrm{Add}(\omega, 1)$ is c.c.c. and hence preserves all cardinals and the stationarity of subsets of uncountable regular cardinals.

We will use a fact that follows immediately from Theorem 7.1 and Theorem 5.3 of [4].

Fact 3.4. Suppose $2^{\aleph_0} < \mu_0$. If $\mathbb{P}$ is c.c.c. and G is generic for $\mathbb{P}$ over V. If $M \in V[G]$ is tight for $\langle \mu_n : n < \omega \rangle$ and $\mathrm{cf}(M \cap \mu_n) = \omega_1$ for all $n < \omega$, then $\chi_M \in V$.

By a density argument, for any $f \in V \cap \prod_n \mu_n$, there are infinitely many $n < \omega$ so that $f(n) \notin S^{x(n)}_n$. Therefore, no tight structure can meet $\langle S^{x(n)}_n : n < \omega \rangle$, and hence it is not tightly stationary. □

Theorem 3.5. *Suppose* $2^{\aleph_0} < \aleph_\omega$. *In the forcing extension by* $\mathrm{Add}(\omega, \omega_1)$, *for any sequence of cardinals* $\langle \mu_n : n < \omega \rangle$ *and any stationary sequence* $\langle T_n : n < \omega \rangle$ *on* $\langle \mu_n : n < \omega \rangle$ *with* $T_n \in V$ *for each* n, *there is a stationary sequence* $\langle S_n : n < \omega \rangle$ *which is not tightly stationary, with* $S_n \subseteq T_n$ *for all* n .

Proof. Let $\langle \dot{\mu}_n : n < \omega \rangle$ be a name for an increasing sequence of regular cardinals below κ, and suppose that there is a condition $p \in \mathrm{Add}(\omega, \omega_1)$ which forces that $\langle \dot{\mu_n} : n < \omega \rangle$ has no stationary, not tightly stationary sequence.

For each $n < \omega$, let $A_n = \{a_n^i : i < \omega\}$ be a maximal antichain in $\mathrm{Add}(\omega, \omega_1)$ so that each a_n^i forces a value for μ_n. Since $\mathrm{Add}(\omega, \omega_1)$ is c.c.c., each A_n is countable and hence $\gamma := \sup(\bigcup_{n,i} \mathrm{dom}(a_n^i)) < \omega_1$. So $\langle \mu_n : n < \omega \rangle \in V^{\mathrm{Add}(\omega,\gamma)}$, where $\mathrm{Add}(\omega, \gamma)$ is thought of as an initial segment of $\mathrm{Add}(\omega, \omega_1)$. Since $\mathrm{Add}(\omega, \omega_1)$ factors as $\mathrm{Add}(\omega, \gamma) \times \mathrm{Add}(\omega, 1) \times \mathrm{Add}(\omega, \omega_1)$, the result follows from Proposition 3.3. $\square$

We remark that the previous result is quite indestructible. For example, if H is generic for $\mathrm{Add}(\omega, \omega_1)$ and $\mathbb{P}$ is Prikry forcing over $V[H]$, then the Levy–Solovay theorem says that the normal ultrafilter U used by $\mathbb{P}$ is in fact generated by a normal ultrafilter $\bar{U}$ in V. It is then easy to check that over V, $\mathrm{Add}(\omega, \omega_1) * \mathbb{P}$ is forcing equivalent to $\bar{\mathbb{P}} \times \mathrm{Add}(\omega, \omega_1)$, where $\bar{\mathbb{P}}$ is Prikry forcing over V using $\bar{U}$, and therefore in the further Prikry extension, every sequence of regular cardinals carries a stationary sequence which is not tightly stationary.

This gives another method to obtain mutually stationary but not tightly stationary sets, since on any Prikry sequence, every stationary sequence is mutually stationary (Theorem 5.4 of [5], using our slightly weaker definition for mutual stationarity). A similar method works for increasing sequences of measurable cardinals. Compared to the examples of mutually stationary but not tightly stationary sequences in [5], these examples have the disadvantage of requiring large cardinals. However, they are more flexible in the ways that they can be iterated, and this flexibility can be used to obtain results of a global nature as in Theorem 5.7.

4. Scales in the Prikry extension

There are perhaps three natural approaches for forcing a positive answer to Question 1.1: (1) destroying the mutual stationarity of a sequence which is not tightly stationary, (2) making a mutually stationary sequence tightly stationary, or (3) forcing to add a new sequence of regular cardinals for which every mutually stationary sequence is tightly stationary.

The first approach appears to be quite difficult. For example, suppose $\vec{S}$ is a mutually stationary sequence on a sequence of regular cardinals with limit μ, and $\mathbb{P}$ is a μ-c.c. forcing notion. Since μ is singular, in fact $\mathbb{P}$ is ν-c.c. for some $\nu < \mu$. So for any function $F : [\mu]^{<\omega} \to \mu$ in the extension by $\mathbb{P}$, there is a function $\hat{F} : [\mu]^{<\omega} \to \mu$ in the ground model so that any set closed under $\hat{F}$ which contains ν as a subset is also closed under F. Therefore $\vec{S}$ remains mutually stationary after forcing with $\mathbb{P}$.

Since tight stationarity is connected with the continuous scales on the sequence of regular cardinals, the second approach would involve forcing so that ground model scales are not cofinal (or even just adding new sequences of regular cardinals altogether, as in the third approach). Variants of Prikry forcing are essentially the only techniques known for achieving this. Analysis of the scales on certain products after Prikry-type forcing was done by Jech [12] (ordinary Prikry forcing), Cummings–Foreman [3] (supercompact diagonal forcing), and Lambie-Hanson [17] (supercompact diagonal forcing). Below, we give a slight generalization of the result in [12] for the ordinary Prikry forcing.

Let κ be a measurable cardinal, and $j : V \to M$ the ultrapower embedding by a normal measure U on κ. Let $\mathbf{Pr}$ be the Prikry forcing using U. Recall that this is the set of all pairs (s, A) with s a finite increasing sequence from κ and $A \in U$, ordered by $(s, A) \leq (s', A')$ if and only if $s \supseteq s'$, $A \subseteq A'$ and $s \setminus s' \subseteq A'$. There is also an auxiliary ordering $\leq^*$ defined by $(s, A) \leq^* (s', A')$ if and only if $s = s'$ and $A \subseteq A'$. We abuse notation and identify a generic E for $\mathbf{Pr}$ with the ω-sequence $\langle \zeta_n : n < \omega \rangle$ it adds. Let LP (for "lower part") be the set of all finite increasing sequences from κ.

There is one crucial feature of Prikry forcing, which can be used to prove many properties of the generic extension, for example that $\mathbf{Pr}$ does not add bounded subsets of κ.

Fact 4.1.
(1) For every dense open $D \subseteq \mathbf{Pr}$ and $(s, A) \in \mathbf{Pr}$, there is $n < \omega$ and $(s, A^*) \leq^* (s, A)$ so that any $(s', A') \leq (s, A^*)$ with $|s'| \geq n$ is in D.
(2) As a consequence, we have the *Prikry property*: for any $(s, A) \in \mathbf{Pr}$ and any statement φ in the forcing language, there is $(s, A^*) \leq^* (s, A)$ so that φ is decided by (s, A^*).

These are proven using a diagonal intersection argument. A particularly convenient way of taking diagonal intersections in Prikry forcing can be expressed when there is a sequence of measure one sets $\langle A_s : s \in \mathrm{LP} \rangle$, and we define the diagonal intersection to be

$$\Delta_s A_s = \Delta_\xi \bigcap_{s : s(|s|)-1) = \xi} A_s.$$

The following characterization of genericity for $\mathbf{Pr}$ is due to Mathias.

Fact 4.2. A sequence $\langle \zeta_n : n < \omega \rangle$ is generic for $\mathbf{Pr}$ if and only if for each $A \in U$, there is $n_A < \omega$ so that $\zeta_n \in A$ for all $n \geq n_A$.

We will make use of the iterated ultrapowers M_n for $n \leq \omega$. These are defined recursively, together with a commuting system of elementary embeddings $i_{m,n} : M_m \to M_n$ for $m < n$.

(1) $M_0 = V$ and $i_{0,1} = j$.
(2) $M_{n+1} = \mathrm{Ult}(M_n, i_{0,n}(U))$ and $i_{n,n+1}$ is the ultrapower embedding.
(3) $i_{m,n+1} = i_{n,n+1} \circ i_{m,n}$ for $m < n$.
(4) M_ω is the direct limit of the system of ultrapowers $\langle M_n, (i_{m,n}) \rangle$, and the maps $i_{m,\omega}$ are the direct limit embeddings.

We list some basic facts about the iterated ultrapowers. For notational simplicity, we write $\kappa_n := i_{0,n}(\kappa)$.

Fact 4.3. Let $\langle M_n, (i_{m,n}) \rangle$ be the system of iterated ultrapowers defined above.
(1) $i_{0,n} = j^n$ (that is, j composed with itself n times).
(2) Every member of M_n can be written as $i_{0,n}(G)(\kappa_0, \kappa_1, \ldots, \kappa_{n-1})$ for some $G : [\kappa]^n \to V$.

Bukovský and Dehornoy [7] independently proved a connection between the generic extension by Prikry forcing and the iterated ultrapowers.

Fact 4.4. (1) The sequence $\langle \kappa_0, \kappa_1, \ldots \rangle$ is generic for $i_{0,\omega}(\mathbf{Pr})$ over M_ω.
(2) The generic extension $M_\omega[\langle \kappa_0, \kappa_1, \ldots \rangle]$ is equal to the intersection of the M_n, $n < \omega$.

Motivated by this result, we write $N_\omega := \bigcap_{n<\omega} M_n$. We will use the fact that N_ω has the same ω-sequences of ordinals as V.

Definition 4.5. If c is a $\mathbf{Pr}$-name, define $[\![c]\!]$ to be the evaluation of $i_{0,\omega}(c)$ using the M_ω-Prikry-generic sequence $\langle \kappa_0, \kappa_1, \ldots \rangle$.

The definition also relativizes to $\mathbf{Pr}$ below some condition (s, A). Note that N_ω is also the Prikry extension of M_ω using the generic sequence $s^\frown \langle \kappa_0, \kappa_1, \ldots \rangle$.

Definition 4.6. If $s \in \mathrm{LP}$, then for a $\mathbf{Pr}$-name c, define $[\![c]\!]_s$ to be the evaluation of $i_{0,\omega}(c)$ using the M_ω-Prikry-generic sequence $s^\frown \langle \kappa_{|s|}, \kappa_{|s|+1}, \ldots \rangle$.

Lemma 4.7. *Let ϕ be a formula in the language of set theory, and $\{c_i : i < n\}$ be $\mathbf{Pr}$-names. If for every $s \in \mathrm{LP}$ we have*

$$N_\omega \vDash \phi([\![c_0]\!]_s, \ldots, [\![c_n]\!]_s),$$

then $\Vdash_{\mathbf{Pr}} \phi(c_0, \ldots, c_n)$.

Proof. For each $s \in \mathrm{LP}$, there is a set $A_s \in U$ so that (s, A_s) decides $\phi(c_0, \ldots, c_n)$. In fact, $(s, A_s) \Vdash \phi(c_0, \ldots, c_n)$; otherwise $(s, A_s) \Vdash \neg\phi(c_0, \ldots, c_n)$ so

$$(s, i_{0,\omega}(A_s)) \Vdash \neg\phi(i_{0,\omega}(c_0), \ldots, i_{0,\omega}(c_n)).$$

But $\kappa_m \in i_{0,\omega}(A_s)$ for each $m < \omega$, so $(s, i_{0,\omega}(A_s))$ is compatible with the M_ω-Prikry-generic sequence $s^\frown \langle \kappa_{|s|}, \kappa_{|s|+1}, \ldots \rangle$, so $N_\omega \vDash \neg\phi([\![c_0]\!]_s, \ldots, [\![c_n]\!]_s)$, contradiction.

Take $A^* = \Delta_s A_s$. Any condition (t, B) can be strengthened to $(t, B \cap A^*)$, and any further strengthening of this must be compatible with (t, A_t), so we have shown that the set of conditions forcing $\phi(c_0, \ldots, c_n)$ is dense. □

Suppose that $\langle \mu_n : n < \omega \rangle$ is an increasing sequence of regular cardinals in $V[E]$ cofinal in κ. Using Fact 4.1, we can find a name for $\langle \mu_n : n < \omega \rangle$ of a particularly nice form.

Lemma 4.8. *In V, there is a name $\langle \dot\mu_n : n < \omega \rangle$ for $\langle \mu_n : n < \omega \rangle$ so that:*
(1) There are $\sigma : \omega \to \omega$ (the arity function) and $F^n : [\kappa]^{\sigma(n)} \to \kappa$ so that for any n, if $|s| = \sigma(n)$, then $\langle s, \kappa \rangle$ forces $\dot\mu_n = F^n(s)$.
(2) Every ordinal in the image of F^n is a regular cardinal.
(3) There is a non-decreasing, unbounded function $\rho : \omega \to \omega \cup \{-1\}$ so that $F^n(\xi_0, \ldots, \xi_{\sigma(n)-1}) > \xi_{\rho(n)}$ for all $(\xi_0, \ldots, \ldots, \xi_{\sigma(n)-1}) \in [\kappa]^{\sigma(n)}$, where we define $\xi_{-1} = 0$.

We will call such a name *normal.*

For each n, $0 < n < \omega$, and each $\gamma < \kappa_n$, fix functions $G^n_\gamma : [\kappa]^{\sigma(n)} \to \kappa$ such that $i_{0,n}(G^n_\gamma)(\kappa_0, \kappa_1, \ldots, \kappa_{n-1})) = \gamma$, i.e., G^n_γ represents γ in the nth iterated ultrapower.

Define a function τ on κ_ω by setting $\tau(\gamma)$ to be largest so that $\gamma \in \mathrm{image}(i_{0,\tau(\gamma)})$.

Lemma 4.9. *Suppose $G : [\kappa]^n \to \kappa$ such that*

$$i_{0,n}(G^n_\gamma)(\kappa_0, \kappa_1, \ldots, \kappa_{n-1}) \in \mathrm{image}(i_{0,m}).$$

Then there is $A \in U$ such that G restricted to A does not depend on the first m coordinates of the input, i.e., for all sequences $x_1 < \ldots < x_{m-1}$ and $y_1 < \ldots < y_{m-1}$, and $z_m < \ldots < z_n$ with $x_{m-1}, y_{m-1} < z_m$,

$$G(x_1, \ldots, x_{m-1}, z_m, \ldots, z_{n-1}) = G(y_1, \ldots, y_{m-1}, z_m, \ldots, z_{n-1}).$$

So by changing the functions G^n_γ on a measure zero set, we may assume that G does not depend on the first $\tau(\gamma)$ coordinates of the input. From now on, we enforce this assumption.

For certain sequences of regular cardinals, scales on κ in the Prikry extension are closely related to ground model scales on (ground model) singular cardinals. Note that the Prikry forcing cannot add new scales to singular cardinals other than κ: it does not add any ω-sequences bounded below κ, and it has the κ-c.c. so cannot add an ω-sequence unbounded by ground model ω-sequences in the product of regular cardinals above κ.

Definition 4.10. A normal name given by $\langle F^n : n < \omega \rangle$ with $\mathrm{dom}(F^n) = [\kappa]^{\sigma(n)}$ is *forgetful* if there is a non-decreasing, unbounded function $\tau : \omega \to \omega$ so that F^n does not depend on the first $\tau(n)$ coordinates.

Forgetfulness will be applied through the following straightforward lemma.

Lemma 4.11. *Suppose $\langle F^n : n < \omega \rangle$ is a forgetful normal name and $s \in \mathrm{LP}$. Then there is $n_0 < \omega$ so that for all $n \geq n_0$,*

$$[\![F^n]\!]_s = [\![F^n]\!].$$

Suppose that $\langle \mu_n : n < \omega \rangle \in V[E]$ is an increasing sequence of regular cardinals cofinal in κ. Let F^n, σ, ρ be as in Lemma 4.8, and τ be as in Definition 4.10. We will focus on the special case where $\langle \dot{\mu}_n : n < \omega \rangle$ is forgetful. The theorem will show that in these cases, the pcf structure of $\prod_n \mu_n$ reflects that of $\prod_n [\![\mu_n]\!]$.

Define $k(n) := \min\{\tau(n), \rho(n), \sigma(n)\}$. Since τ,ρ, and σ are non-decreasing and unbounded in ω, so is k.

Lemma 4.12. *For each n, the set of ordinals in $\mathrm{image}(i_{0,k(n)}) \cap [\![\dot{\mu}_n]\!]$ is $< \kappa$-closed and unbounded.*

Proof. Fix $G^n < F^n$. Define H^n to be the function

$$(\xi_0, \ldots, \xi_{\sigma(n)-1}) \mapsto \sup_{\alpha_0, \ldots, \alpha_{k(n)-1} < \xi_{k(n)}} G^n(\alpha_0, \ldots, \alpha_{k(n)-1}, \xi_{k(n)}, \ldots, \xi_{\sigma(n)-1}).$$

Clearly $G^n \leq H^n$.

We now show

$$H^n(\alpha_0, \ldots, \alpha_{k(n)-1}, \xi_{k(n)}, \ldots, \xi_{\sigma(n)-1}) < F^n(\alpha_0, \ldots, \alpha_{k(n)-1}, \xi_{k(n)}, \ldots, \xi_{\sigma(n)-1}).$$

Since $k(n) \leq \tau(n)$, F^n doesn't depend on the first $k(n)$ coordinates, and since $k(n) \leq \rho(n)$, $\xi_{k(n)} < F^n(\xi_0, \ldots, \xi_{\sigma(n)-1})$, so $H^n < F^n$ as $F^n(\xi_0, \ldots, \xi_{\sigma(n)-1})$ is a regular cardinal. So $\mathrm{image}(i_{0,k(n)}) \cap [\![\dot{\mu}_n]\!]$ is unbounded in $[\![\dot{\mu}_n]\!]$.

Since j is continuous at points of cofinality different from κ, $\mathrm{image}(i_{0,k(n)}) \cap [\![\dot{\mu}_n]\!]$ is $< \kappa$-closed. □

For each n, let $C_n = \mathrm{image}(i_{0,k(n)}) \cap [\![\dot{\mu}_n]\!]$.

Theorem 4.13. *Suppose $\langle \dot\mu_n : n < \omega\rangle$ is an ω-sequence of regular cardinals in $V[E]$ with a forgetful normal name. If $\prod_n C_n$ carries a scale $\langle f_\alpha : \alpha < \lambda\rangle$ in V, then in $V[E]$ there is a scale $\langle g_\alpha : \alpha < \lambda\rangle$ on $\prod_n \mu_n$ defined by*

$$g_\alpha(n) = G^{\sigma(n)}_{f_\alpha(n)}(\zeta_0, \ldots, \zeta_{\sigma(n)-1}).$$

If $\langle f_\alpha : \alpha < \lambda\rangle$ is continuous at a point δ, then so is $\langle g_\alpha : \alpha < \lambda\rangle$.

Proof. For each $\alpha < \lambda$, let $\dot g_\alpha$ be the forgetful normal name for g_α given by $\langle G^{\sigma(n)}_{f_\alpha(n)}(\zeta_0, \ldots, \zeta_{\sigma(n)-1}) : n < \omega\rangle$. For any $s \in \mathrm{LP}$, $[\![\dot g_\alpha]\!] =^* [\![\dot g_\alpha]\!]_s$ by Lemma 4.11. Therefore

$$[\![\dot g_\beta]\!]_s <^* [\![\dot g_\beta]\!]_s.$$

By Lemma 4.7, $g_\alpha <^* g_\beta$ in $V[E]$.

Now we check that $\langle g_\alpha : \alpha < \lambda\rangle$ is cofinal in $\prod_n \mu_n$. Suppose that h is a function in $\prod_n \mu_n$ in $V[E]$, and let $\dot h$ be a name for h which is forced to be in $\prod_n \dot\mu_n$. Fix $s \in \mathrm{LP}$ arbitrary. By Lemma 4.7, $[\![h(n)]\!]_s < [\![\dot\mu_n]\!]_s$ for all $n < \omega$. By Lemma 4.11, $[\![\dot\mu_n]\!] =^* [\![\mu_n]\!]_s$. Therefore $[\![h(n)]\!]_s < [\![\dot\mu_n]\!]$ for all but finitely many n. Since $\langle f_\alpha : \alpha < \lambda\rangle$ is a scale on $\prod_n [\![\dot\mu_n]\!]$, there is some $\alpha < \lambda$ so that $[\![h]\!]_s <^* f_\alpha = [\![\dot g_\alpha]\!]$. But $[\![\dot g_\alpha]\!] =^* [\![\dot g_\alpha]\!]_s$, so $[\![h]\!]_s <^* [\![\dot g_\alpha]\!]_s$. As s was arbitrary, $h <^* g_\alpha$ in $V[E]$.

The proof for the continuity part of the theorem is similar. Suppose that h is a function in $\prod_n g_\delta(n)$ in $V[E]$, and let $\dot h$ be a name for h which is forced to be in $\prod_n \dot g_\delta(n)$. Fix $s \in \mathrm{LP}$ arbitrary. Applying Lemma 4.7 and Lemma 4.11 as before, $[\![h(n)]\!]_s < [\![\dot g_\delta(n)]\!]$ for all but finitely many n. Since $\langle f_\alpha : \alpha < \lambda\rangle$ is continuous at δ, there is some $\alpha < \delta$ so that $[\![h]\!]_s <^* [\![\dot g_\alpha]\!] =^* [\![\dot g_\alpha]\!]_s$, so $[\![h]\!]_s <^* [\![\dot g_\alpha]\!]_s$. This implies that $h <^* g_\alpha$ in $V[E]$. □

Finally, we can apply the analysis of these scales to tight stationarity.

Theorem 4.14. *Suppose $\langle \mu_n : n < \omega\rangle \in V[E]$ is given by a forgetful normal name and $\prod_n C_n$ carries a scale $\langle f_\alpha : \alpha < \lambda\rangle$ in V. Then there is a stationary but not tightly stationary sequence $\langle S_n : n < \omega\rangle$.*

Proof. Let $\langle F^n\rangle$, σ, τ, and k be defined for the forgetful normal name for $\langle \mu_n : n < \omega\rangle \in V[E]$ as in Lemma 4.8 and Definition 4.10. Let η be the fixed uncountable cofinality to which we are restricted. Fix a name $\langle \dot g_\alpha : \alpha < \lambda\rangle$ for a scale on $\prod_n \mu_n$ as in Theorem 4.13.

We will construct a name for S_n for each $n < \omega$. By restricting to a final segment, we may assume that $k(n) > 0$. For each $(\xi_{\rho(n)}, \ldots, \xi_{\sigma-1}) \in [\kappa]^{\sigma(n)-k(n)}$, partition $F^n(\xi_0, \ldots, \xi_{\sigma-1}) \cap \mathrm{Cof}(\eta)$ into $\xi_{k(n)}$ disjoint stationary sets, noting that F^n does not depend on its first $k(n)$ arguments and $\xi_{k(n)} \leq F(\xi_0, \ldots, \xi_{\sigma-1})$. Let $T_n^{(\xi_{kn}, \ldots, \xi_{\sigma(n)-1})}$ be an injection from $\xi_{k(n)}$ into the collection of these stationary sets. Define $S_n : [\kappa]^{\sigma(n)} \to \kappa$ by

$$S_n(\xi_0, \ldots, \xi_{\sigma(n)-1}) = T_n^{(\xi_{k(n)}, \ldots, \xi_{\sigma(n)-1})}(\xi_{k(n)-1}).$$

This function gives a name for the set $S_n \in V[E]$.

We will show that in $V[E]$, $\nu(\langle S_n\rangle) = \emptyset$, which is stronger than required by Lemma 2.2 to show that $\langle S_n : n < \omega\rangle$ is not tightly stationary. Suppose otherwise, so there is $(s, A) \in \mathbf{Pr}$ and $\alpha < \lambda$ so that (s, A) forces $\alpha \in \nu(\langle \dot S_n\rangle)$. By extending (s, A) we may assume that there is some fixed $n < \omega$ with $k(n) > |s|$ so that (s, A) forces $\dot g_\alpha(n) \in \dot S_n$.

Take $\xi_0, \ldots, \xi_{\sigma(n)-1} \in [A]^{\sigma(n)-k(n)}$ so that $\xi_i = s(i)$ for $i < |s|$ and $A \cap (\xi_{k(n)-1}, \xi_{k(n)}) \neq \emptyset$. Choose $\xi'_{k(n)-1} \in A \cap (\xi_{k(n)-1}, \xi_{k(n)})$. Let us write $\bar{\xi}$ for $\xi_0, \ldots, \xi_{\sigma(n)-1}$ and $\bar{\xi}'$ for the sequence obtained from $\bar{\xi}$ by replacing $\xi_{k(n)-1}$ with $\xi'_{k(n)-1}$.

Since (s, A) forces $\dot{g}_\alpha(n) \in \dot{S}_n$,

$$G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}) \in S_n(\bar{\xi}).$$

By the same reasoning,

$$G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}') \in S_n(\bar{\xi}').$$

Since $\bar{\xi}$ and $\bar{\xi}'$ differ only in the $k(n)-1$ coordinate, and $G^{\sigma(n)}_{f_\alpha(n)}$ does not depend on its first $k(n)$ arguments, $G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}) = G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}')$. Therefore,

$$G^{\sigma(n)}_{f_\alpha(n)}(\bar{\xi}) \in S_n(\bar{\xi}) \cap S_n(\bar{\xi}'),$$

but this is impossible as $S_n(\bar{\xi})$ and $S_n(\bar{\xi}')$ were chosen to be disjoint. □

Still, we do not know what happens when the sequence is not forgetful; see Section 6 for some consistency results.

5. Mutually stationary but not tightly stationary sequences on every sequence

In this section, we will construct a model where every increasing ω-sequence of regular cardinals has a mutually stationary sequence which is not tightly stationary.

First, we will show that for a sequence of regular cardinals with interleaved supercompacts, every stationary sequence is mutually stationary. Our argument is a supercompact version of the proof of Theorem 5.2 in [5].

Suppose κ is λ-supercompact, and U is a normal, fine ultrafilter on $[\lambda]^{<\kappa}$.

For any n and $x, y \in [\lambda]^{<\kappa}$, say that $x \subsetneq y$ if $|x| < |y \cap \kappa|$. Say that $x < y$ if $x \subsetneq y$ and $x \in \mathrm{Sk}(y)$, where the Skolem hull is computed in the structure $(H(\lambda), \in, \lhd)$. Here $\lhd$ is a fixed well-ordering of $H(\theta)$ (this is a standard device useful for making things definable without parameters).

Supercompactness measures satisfy the following partition property (for a reference, see Kanamori [14]):

Fact 5.1. For any $n < \omega$ and $f : ([\lambda]^{<\kappa})^n \to 2$, there is $Y \in U$ homogeneous for f, i.e., there is $i \in 2$ so that $f(x_0, x_1, \ldots, x_{n-1}) = i$ for any $x_0 < x_1 < \cdots < x_{n-1}$ all from Y.

Proposition 5.2. *Suppose $\langle \lambda_n : n < \omega \rangle$ is an increasing sequence of regular cardinals and $\langle \kappa_n : n < \omega \rangle$ is a sequence of cardinals so that for each $n < \omega$,*

(1) κ_n is λ_n-supercompact,
(2) $\lambda_{n-1} < \kappa_n \leq \lambda_n$,
(3) $\zeta^\xi < \lambda_n$ for cardinals $\xi < \kappa_n$ and $\zeta < \lambda_n$.

Then any sequence $\langle S_n : n < \omega \rangle$ with $S_n \subseteq \lambda_n \cap \mathrm{Cof}(< \kappa_n)$ is mutually stationary.

Proof. For each $0 < n < \omega$, let U_n be a normal, fine ultrafilter on $[\lambda_n]^{<\kappa_n}$.

Let $\mathcal{A}$ be an arbitrary expansion of $(H(\theta); \in, \lhd, \langle \lambda_n, \kappa_n, U_n : n < \omega \rangle)$ for $\theta = \sup_n \lambda_n$. For each n with $0 < n < \omega$, U_n concentrates on the closed unbounded set X_n of structures x so that

(1) $\lambda_{n-1} \subseteq x$,
(2) $x \cap \kappa_n \in \kappa_n$,
(3) $\mathrm{Sk}^{\mathcal{A}}(x) \cap \lambda_n = x$.

Notice that this condition implies that $X_m \cap X_n = \emptyset$ for $m \neq n$.

Given a finite $<$-increasing $\bar{x} \subset \bigcup_i X_i$, define the *type* of $\bar{x}$ to be the function $n \mapsto |\bar{x} \cap X_n|$. If $a \subseteq \omega$, then we define $t \restriction a$ to be the type which is equal to t on a, and takes value 0 elsewhere.

We will construct $\langle Y_n : 0 < n < \omega \rangle$ so that:

(1) For each n, $Y_n \subseteq X_n$ and $Y_n \in U_n$,
(2) (Indiscernibility) If φ is a formula in the language of $\mathcal{A}$ using only ordinal parameters $\bar{c} \subseteq \sup_n \lambda_n$, and $\bar{x}$, $\bar{y}$ are finite $<$-increasing sequences from $\bigcup_i Y_i$ of the same type which fit as the free variables of φ so that

$$\{x \in \bar{x} : \bar{c} \notin x\} = \{y \in \bar{y} : \bar{c} \notin y\},$$

then

$$\varphi(\bar{c}, \bar{x}) \text{ iff } \varphi(\bar{c}, \bar{y}).$$

Fix a formula φ and parameters $\bar{c}$ and a type t with $\sum_i t(i)$ equal to the number of free variables of φ. We will construct $\langle Y_n^{\varphi,t,\bar{c}} : 0 < n < \omega \rangle$ that works for φ, t, and $\bar{c}$, and then set $Y_n = \bigcap_{\varphi,t} \Delta_{\bar{c}} Y_n^{\varphi,t,\bar{c}}$ (we abuse notation in this diagonal intersection by identifying $\bar{c}$ with its ordinal code under some coding of finite tuples definable over $\mathcal{A}$ and closed at inaccessible cardinals).

Let n_t be the least $n < \omega$ so that $t(n') = 0$ for all $n' \geq n$. If $n > n_t$, then set $Y_n^{\varphi,t,\bar{c}}$ equal to X_n.

Now by induction on $m < n_t$ we define $Y_{n_t-m}^{\varphi,t,\bar{c}} \in U_{n_t-m}$ so that if $\bar{x}, \bar{y}$ are of type t with:

(1) $\bar{x} \cap X_n \subseteq Y_n^{\varphi,t,\bar{c}}$ for all $n \geq n_t - m$
(2) $\bar{x} \cap \bigcup_{i<n_t-m} X_i = \bar{y} \cap \bigcup_{i<n_t-m} X_i$,

then $\varphi(\bar{c}, \bar{x})$ iff $\varphi(\bar{c}, \bar{y})$.

So fix $m < n_t$ and suppose that $Y_n^{\varphi,t,\bar{c}}$ has already been defined for $n > n_t - m$. For each $\bar{v} \subseteq \bigcup_{n<n_t-m} X_n$ of type $s \restriction (n_t - m)$, let $Y_{n_t-m}^{\varphi,t,\bar{c},\bar{v}} \in U_{n_t-m}$ be homogeneous for the function mapping a length $t(n_t - m)$-sequence $\bar{u}$ in $[\lambda_{n_t-m}]^{<\kappa_{n_t-m}}$ to the truth value of $\varphi(\bar{c}, \bar{v}, \bar{u}, \bar{u}_{\text{upper}})$, where $\bar{u}_{\text{upper}}$ is a sequence of the appropriate type from $\bigcup_{n>n_t-m} Y_n^{\varphi,t,\bar{c}}$. By the induction hypothesis, this does not depend on the choice of $\bar{u}_{\text{upper}}$. Then define $Y_{n_t-m}^{\varphi,t,\bar{c}} = \bigcap_{\bar{v}} Y_{n_t-m}^{\varphi,t,\bar{c},\bar{v}}$. This completes the construction.

Suppose $n < \omega$. For each $x \in [\lambda_n]^{<\kappa_n}$ and $\xi < \lambda_n$, define $y(x, \xi)$ to be the $\lhd$-least structure $y > x$ in Y_n containing ξ as an element. By our cardinal arithmetic assumptions, the function $\xi \mapsto \sup_{x \subseteq \xi}\{\sup y(x, \xi)\}$ maps λ_n into λ_n, so the closure points form a club C_n. Therefore, we can take $I_n \subseteq Y_n$ which is $<$-increasing of limit order-type so that $\gamma_n := \sup \bigcup I_n \in S_n$ for each $0 < n < \omega$.

Finally set $W = \mathrm{Sk}^{\mathcal{A}}(\bigcup_i I_i)$. It remains to check that $\sup(W \cap \lambda_n) = \gamma_n$ for each n. Suppose $\delta \in W \cap \lambda_n$ for some n. Then $\delta = t(\bar{z})$ for some $\mathcal{A}$-term t and finite $\bar{z} \subseteq \bigcup_i I_i$.

Let $\bar{z}' = \bar{z} \cap \bigcup_{i \leq n} X_i$. By indiscernibility, for any $<$-increasing sequence $\bar{u}$ from $\bigcup_n Y_n$ of the same type as $\bar{z} \setminus \bar{z}'$,

$$\delta = t(\bar{z}', \bar{u}).$$

Therefore δ can be defined over $\mathcal{A}$ with parameter $\bar{z}'$ as "the unique ordinal for which there exist measure one sets so that $\delta = t(\bar{z}', \bar{u})$ whenever $\bar{u}$ is an increasing sequence of the right type taken from those measure one sets." Now take $x \in I_n$ with $x > z$ for every $z \in \bar{z}'$. Since $\mathrm{Sk}^{\mathcal{A}}(x)$ contains $\bar{z}'$, $\delta \in x$, so $\delta < \sup(x) < \sup \bigcup I_n$. □

Koepke [15] adapted the argument from [5] to work for cardinals which were formerly measurable but have been collapsed by forcing. Thus, he was able to force to get a mutual stationarity property, for example, on the sequence $\langle \aleph_{2n+1} : n < \omega \rangle$ (note that there is a gap between successive members of this sequence). We can adapt our Proposition 5.2 using his methods, and combine this with Theorem 3.5 to get a global result on the existence of mutually stationary, not tightly stationary sequences.

The gap between successive cardinals in the sequence seems to be crucial to this argument. For example, Koepke and Welch [16] showed that:

(1) the existence of a measurable cardinal is equiconsistent with the statement that there is some sequence of regular cardinals where every stationary sequence concentrating on ordinals of cofinality ω_1 is mutually stationary, and
(2) to have that every sequence of stationary sets on $\langle \aleph_n : n < \omega \rangle$ concentrating on ordinals of cofinality ω_1 is mutually stationary requires an inaccessible limit of measurable cardinals (and no upper bound is currently known).

The first theorem of this section is a prototype for those which follow.

Theorem 5.3. *If there is a proper class of supercompact cardinals, then there is a class forcing extension so that for every increasing ω-sequence of regular cardinals $\langle \lambda_n : n < \omega \rangle$ so that for each n the interval $(\lambda_n, \lambda_{n+1})$ contains at least three cardinals, any stationary sequence on cofinality ω_1 is mutually stationary.*

Proof. The basic strategy of the proof will be to start from a proper class of supercompact cardinals and force with collapsing posets preserving only

(1) cardinals from the given class,
(2) limits of cardinals from the class,
(3) and ground model successors of cardinals of either type (1) or (2).

We will use an argument adapted from [15] to show that for any increasing ω-sequence $\langle \lambda_n : n < \omega \rangle$ of regular cardinals in the extension so that for each $n < \omega$ there is a formerly supercompact cardinal κ_n so that $\lambda_{n-1} < \kappa_n < \lambda_n$, every stationary sequence of cofinality ω_1 ordinals is mutually stationary. Koepke's result is an equiconsistency with the existence of a measurable cardinal, but certain aspects of his proof simplify in our case using the stronger large cardinal assumptions.

By doing some preliminary forcing if necessary, assume Martin's Axiom $\mathsf{MA}(\aleph_1)$, $2^{\aleph_0} < \aleph_\omega$, and GCH above ω_2. Force so that there is a proper class of indestructibly supercompact cardinals and GCH is preserved at these supercompact cardinals (this also preserves Martin's Axiom). Let $\mathsf{SC} = \langle \mu_\xi : \xi < \mathsf{ON} \rangle$ be a continuous increasing sequence so that $\mu_0 = \omega_1$ and μ_ξ is one of these indestructibly supercompacts for every successor ordinal ξ. Let $\mathbb{P}$ be the class length Easton support product of the posets $\mathbb{Q}_{\mu_\xi} := \mathrm{Col}^V(\mu_\xi^+, < \mu_{\xi+1})$ for each $\mu_\xi \in \mathsf{SC}$. The final model will be a model of ZFC, see Jech [13] for details on class forcing.

Let G be generic for $\mathbb{P}$. Write $G \restriction \mu$ for $G \cap \mathbb{P} \restriction \mu$. In the final model the uncountable cardinals are all of the form μ_ξ or μ_ξ^+ for an ordinal ξ.

Suppose $\langle \lambda_n : n < \omega \rangle$ is an increasing sequence of regular cardinals of $V[G]$ with limit $\lambda = \sup_n \lambda_n$. Assuming that there are at least three $V[G]$ cardinals in each interval $(\lambda_n, \lambda_{n+1})$, we can find a sequence of V-supercompact cardinals $\langle \kappa_n : n < \omega \rangle$ so that $\kappa_n = \mu_{\xi_n+1}$ (i.e., κ_n has a successor index in the sequence of supercompacts), and letting $\kappa'_n = \mu_{\xi_n}$,

$$\lambda_{n-1} \leq \kappa'_n < \kappa_n < \lambda_n$$

for all $0 < n < \omega$.

Let $\mathbb{P}^{(n)} = \mathbb{Q}_{\kappa'_n}$, and $\mathbb{R}^{(n)} = \mathbb{P}\restriction\kappa'_n \times \mathbb{P} \setminus \kappa_n$ be the quotient of $\mathbb{P}$ by $\mathbb{P}^{(n)}$. Each κ_n remains supercompact after forcing with $\mathbb{R}^{(n)}$, since for each n this poset factors into the product of a poset which is κ_n-directed closed and a poset of size $< \kappa_n$. Let U_n be a normal fine ultrafilter U_n on $[\lambda_n]^{<\kappa_n}$ in $V[H_n]$.

For $p \in \mathbb{P}^{(n)}$ and $\alpha < \kappa_n$, we will use the notation $p\restriction\alpha$ to denote the condition given by the restriction of p to domain $\mathrm{dom}(p) \cap (\alpha \times (\kappa'_n)^+)$.

We will show that in $V[G]$, any sequence of stationary subsets of $\langle \lambda_n : n < \omega \rangle$ concentrating on cofinality ω_1 is mutually stationary. Let g_n, H_n be the generics for $\mathbb{P}^{(n)}, \mathbb{R}^{(n)}$, respectively, determined by G. Fix $\dot{F}$ a $\mathbb{P}$-name for a function $F : [\lambda]^{<\omega} \to \lambda$ and $\langle \dot{S}_n : n < \omega \rangle$ a $\mathbb{P}$-name for a sequence of stationary sets of cofinality ω_1 points in $\langle \lambda_n : n < \omega \rangle$.

We will find $W \subseteq \lambda$ so that

$$\sup(\dot{F}``[W]^{<\omega} \cap \lambda_n) \leq \sup(W \cap \lambda_n) \in S_n$$

for each $n < \omega$. Since the choice of F was arbitrary, this suffices for mutual stationarity of $\langle S_n : n < \omega \rangle$. We will abuse notation slightly to write $\dot{F}(\bar{x})$ for $\dot{F}(\sup(x_0), \ldots, \sup(x_n))$, where $\bar{x} = (x_0, x_1, \ldots, x_n)$. Let X_n be defined as in the proof of Proposition 5.2, using κ_n, λ_n.

In $V[G]$, let θ be $(2^\lambda)^+$ and

$$\mathcal{A} = (H(\theta); \in, \lhd, G, \langle \lambda_n, \kappa_n, U_n : n < \omega \rangle, \mathbb{P}, \dot{F}, p)$$

and take $\tilde{N} \prec \mathcal{A}$ countable. Define $N = \tilde{N} \cap V$, so $\tilde{N} = N[G]$.

Since N is countable, $N \in V$. Fix a sequence $\langle e_k : k < \omega \rangle$ so that for every $e < \omega$, there are infinitely many k with $e = e_k$. The number e_k will correspond to the arity of a function in the kth step of our construction. We will say that $\langle \sigma_n^k : k < \omega \rangle$ is a *system with stem* $p \in \mathbb{P}^{(n)}$ *and domain* $Y \in U_n$ if $\sigma_n^k : [Y']^{e_k} \to \mathbb{P}^{(n)}$ for some $Y' \supseteq Y$, and $p \leq \sigma_n^k(\bar{x})\restriction(\min(\bar{x}) \cap \kappa_n)$.

Define $Y_n = \bigcap(U_n \cap N[H_n])$, the intersection of all measure one sets in $N[H_n]$. Since $N[H_n]$ is countable, $Y_n \in U_n$.

Let $\langle F_n^j : j < \omega \rangle$ be an enumeration of the functions $[\bigcup_{i \leq n} X_i]^{<\omega} \to \lambda_n$ in $N[G\restriction\lambda_n]$.

Claim 5.4. *For every n, there are $p \in g_n$ and σ^k a system with stem p and domain Y_n satisfying the following properties:*

(1) For every $\tilde{k} < \omega$, every $t > e_{\tilde{k}}$, and any $a_0, a_1 \in [t]^{e_{\tilde{k}}}$, there is $k > \tilde{k}$ so that if $\sigma^{\tilde{k}}(\bar{x}\restriction a_0)$ and $\sigma^{\tilde{k}}(\bar{x}\restriction a_1)$ are compatible, then

$$\sigma^k(\bar{x}) \leq \sigma^{\tilde{k}}(\bar{x}\restriction a_0), \sigma^{\tilde{k}}(\bar{x}\restriction a_1).$$

(2) For every $\tilde{k}$, every $t > e_{\tilde{k}}$, and any $a \in [t]^{e_{\tilde{k}}}$, there is $k > \tilde{k}$ so that $e_k = t$ and $\sigma^k(\bar{x}) \leq \sigma^{\tilde{k}}(\bar{x}\restriction a)$.

(3) *$\sigma^k(\bar{x})$ forces values for $F_n^j(\bar{y}, \bar{z})$ for all $j < k$, all $\bar{y}, \bar{z}$ of the appropriate type with $\bar{y} \subseteq \bigcup_{i<n} X_i$ and $\bar{z} \subseteq \bar{x}$.*

(4) *For each k, $\sigma^k \in N[H_n]$.*

Proof of Claim 5.4. The construction is by induction on k. Fix in advance a suitable bookkeeping so that for every $\tilde{k}, t, a_0, a_1$ as in (1) or (2), there is $k > \tilde{k}$, a stage with the correct arity where we construct to satisfy the corresponding clause. For each $k < \omega$, we will construct in $N[H_n]$ the following objects:

- $p^k \in g_n$,
- $Y^k \in U_n$,
- $\langle \sigma^k(\bar{x}) : |\bar{x}| = e_k \rangle$ which satisfies the conditions required by the bookkeeping and that for all increasing $\bar{x}$ from Y^k,

$$\sigma^k(\bar{x}) \restriction (\min(\bar{x}) \cap \kappa_n) = p^k.$$

Start with some $p^0 \in g_n$ satisfying the demands of (3); this is possible since $\mathbb{P}^{(n)}$ is λ_{n-1}-distributive over $N[H_n]$. Suppose we have already constructed p^i, Y^i, and σ^i for $i \leq k$. We will describe the construction for stage $k+1$.

Working below an arbitrary condition $q \leq p^k$, build conditions $\sigma^{k+1,q}(\bar{x})$ for each $\bar{x}$ with $|\bar{x}| = e_{k+1}$ satisfying the demands of the bookkeeping and of property (3). There are $p^{k+1,q} \in \mathbb{P}^{(n)}$ and $Y^{k+1,q} \in U_n$ so that for any $<$-increasing $\bar{x} \in [Y^{k+1,q}]^{e_{k+1}}$, $\sigma^{k+1,q}(\bar{x}) \restriction (\min(\bar{x}) \cap \kappa_n) = p^{k+1,q}$. The set $D^{k+1} = \{p^{k+1,q} : q \leq p^k\}$ is dense below p^k and a member of $N[H_n]$, so there is $p^{k+1} \in g_n \cap D^{k+1} \cap N$, and we can choose Y^{k+1} and σ^{k+1} to be the corresponding objects in $N[H_n]$.

Let $p_n := \bigcup_k p^k$. By definition of Y_n, $Y_n \subseteq \bigcap_k Y^k$, so p_n is a stem for the system restricted to domain Y_n. This completes the construction. □

Using an argument similar to the proof of Proposition 5.2, Y_n satisfies the following indiscernibility property: if φ is a formula in the language of $\mathcal{A}$ using only ordinal parameters $\bar{c} \subseteq \lambda$, and $\bar{x}$, $\bar{y}$ are finite increasing sequences from Y_n of the same type which fit as the free variables of φ so that

$$\{x \in \bar{x} : \bar{c} \not\subseteq x\} = \{y \in \bar{y} : \bar{c} \not\subseteq y\},$$

then

$$\varphi(\bar{x}) \text{ iff } \varphi(\bar{y}).$$

Define $C_n \subseteq \lambda_n$ to be the club of closure points of the function mapping $\beta < \lambda_n$ to the supremum of the values of $F_n^\ell(\bar{z})$, $\ell < \omega$ and $\sup(\max(\bar{z})) \leq \beta$, forced by some $\sigma_m^k(\bar{x})$ with $m \leq n$, $k < \omega$, and $\sup(\max(\bar{x})) \leq \beta$.

Suppose $I \subseteq Y_n$ is $<$-increasing of order-type ω_1 with $\sup(\bigcup I) \in C_n$. Define $\mathbb{P}_{n,I}$ to be the subposet of $\mathbb{P}^{(n)}$ of conditions of the form $\sigma_n^k(\bar{x})$ for some $k < \omega$ and finite $<$-increasing $\bar{x} \subseteq I$.

Claim 5.5. *The poset $\mathbb{P}_{n,I}$ is c.c.c.*

Proof of Claim 5.5. Suppose otherwise, so there is an uncountable antichain A in $\mathbb{P}_{n,I}$. We can thin to assume that there is a single $k < \omega$ so that $A = \{\sigma_n^k(\bar{z}_i) : i < \omega_1\}$, where the collection of $\bar{z}_i$ forms a Δ-system with root r. There is a uncountable $B \subseteq \omega_1$ so that for every $i \in B$,

$$\max(r \cap X_n) < \min((\bar{z}_i \setminus r) \cap X_n).$$

Pick some $i_0 \in B$. There is an uncountable $B' \subseteq B$ so that for every $i \in B'$, and every $n < \omega$

$$\max(\bar{z}_{i_0} \cap X_n) < \min((\bar{z}_i \setminus r) \cap X_n).$$

Pick some $i_1 \in B'$. Then we claim that $\sigma^k(\bar{z}_{i_0})$ and $\sigma^k(\bar{z}_{i_1})$ are compatible in $\mathbb{P}_I$. By (1) in the construction of $\langle \sigma^k : k < \omega \rangle$, it suffices to check that they are compatible in $\mathbb{P}^{(n)}$.

Let $\beta_0 := \min((\bar{z}_{i_0} \setminus r) \cap X_n) \cap \kappa_n$ and $\beta_1 := \min((\bar{z}_{i_1} \setminus r) \cap X_n) \cap \kappa_n$. Since κ_n is inaccessible, $\sigma^k_n(\bar{z}_{i_0})$ has support bounded in κ_n. There is some $z \in Y_n$ above every member of I so that the support of $\sigma^k_n(\bar{z}_{i_0})$ is a subset of $z \cap \kappa_n$. By indiscernibility, the support of $\sigma^k_n(\bar{z}_{i_0})$ must be a subset of β_1.

Let ρ_n be the $\lhd$-least bijection $\mathbb{P}^{(n)} \to \kappa_n$ with the property that if $p \in \mathbb{P}^{(n)}$ has support which is bounded in an inaccessible β, then $\rho_n(p) < \beta$. Now

$$\gamma := \rho_n(\sigma^k_n(\bar{z}_{i_0}) \restriction \beta_0) < \beta_0,$$

so $\rho_n(\sigma^k_n(\bar{z}_{i_0}) \restriction \beta_0) \in x$ for all $x \in ((\bar{z}_{i_0} \cup \bar{z}_{i_1}) \setminus r) \cap X_n$.

Let $\phi(\bar{z})$ be the statement

$$\gamma = \rho_n(\sigma^k_n(\bar{z}) \restriction \beta_{\bar{z}}),$$

where $\beta_{\bar{z}} = \min((\bar{z} \setminus r) \cap X_n) \cap \kappa_n$. The statement holds for $\bar{z} = \bar{z}_{i_0}$. By indiscernibility it then holds also for $\bar{z} = \bar{z}_{i_1}$. So

$$\sigma^k_n(\bar{z}_{i_0}) \restriction \beta_0 = \sigma^k_n(\bar{z}_{i_1}) \restriction \beta_1.$$

Therefore, $\sigma^k_n(\bar{z}_{i_0}) = \sigma^k_n(\bar{z}_{i_0}) \restriction \beta_1$ is compatible with $\sigma^k_n(\bar{z}_{i_1})$. □

For each $\bar{x} \subseteq I, j < \omega$ define

$$D_{n,\bar{x},j} := \{p \in \mathbb{P}_{n,I} : p \text{ forces a value for } \dot{F}^j_n(\bar{x})\}.$$

The set $D_{n,\bar{x},j}$ is dense in $\mathbb{P}_{n,I}$, since for any $\sigma^k_n(\bar{z}) \in \mathbb{P}_{n,I}$, there is $k' \geq j$ so that $|\bar{z} \cup \bar{x}| = e_{k'}$ and $\sigma^{k'}_n(\bar{z} \cup \bar{x}) \leq \sigma^k_n(\bar{z})$ by condition (2) in the construction of $\langle \sigma^k_n : k < \omega \rangle$, and k'' so that $e_{k''} = e_{k'}$ and $\sigma^{k''}_n(\bar{z} \cup \bar{x}) \leq \sigma^{k'}(\bar{z} \cup \bar{x})$ forces a value for $\dot{F}^j_n(\bar{x})$ by condition (3).

Using Martin's Axiom, there is a generic filter $G_{n,I} \subseteq \mathbb{P}_{n,I}$ which meets each of the $\aleph_1$ many subsets $D_{n,\bar{x},j}$, $\bar{x} \subseteq I$ increasing and $j < \omega$. Define $p_{n,I} := \bigcup G_{n,I}$.

Recall that C_n is the club in λ_n of closure points of the function taking an ordinal ξ to the supremum of values forced by the system applied to $[\xi]^{<\kappa_n}$.

Claim 5.6. *For each $n < \omega$, there is $I_n \subseteq Y_n$ which is $<$-increasing of order-type ω_1 so that $p_{n,I_n} \in g_n$ and $\gamma_n := \sup(\bigcup I_n) \in C_n \cap S_n$.*

Proof of Claim 5.6. Since $|\mathbb{P}^{(n)}| < \lambda_n$, S_n has a stationary subset S'_n in $V[H_n]$ (as there must be a single condition in g_n forcing stationary many ordinals into it). We will show that the set of $p_{n,I}$, I varying among $<$-increasing subsets of Y_n of order-type ω_1 with $\sup(\bigcup I) \in C_n \cap S'_n$, is predense below p_n in $\mathbb{P}^{(n)}$. Since this set belongs to $V[H_n]$, it follows that it is met by g_n, and this implies the claim.

Let $q \leq p_n$ in $\mathbb{P}^{(n)}$ be arbitrary. The elements of $\mathbb{P}^{(n)}$ have domains of size $< \kappa_n$, so there is $\beta < \kappa_n$ with $\operatorname{dom}(q) = \operatorname{dom}(q \restriction \beta)$.

Choose $I \subseteq Y_n$ with $\min(I) \cap \kappa_n > \beta$ and $\sup(\bigcup I) \in S'_n$, which is possible since $Y_n \in U_n$. Since p_n is the stem of the system, $p_n \leq p \restriction (\min(I_n) \cap \kappa_n)$ for any $p \in \mathbb{P}_{n,I}$. So $p_{n,I} \restriction \beta \subseteq p_{n,I} \restriction (\min(I) \cap \kappa_n) \geq p_n$, and therefore $p_{n,I}$ is compatible with q. □

Let $W = \bigcup_n I_n$.

For any $m, n < \omega$, the restriction $F \cap ([\bigcup_{i \leq n} X_i]^{<\omega} \times \lambda_m)$ is a member of $N[G \restriction \lambda_n]$. It therefore suffices to prove that for every $m, n < \omega$, every function $f : [\bigcup_{i \leq n} X_i]^{<\omega} \to \lambda_m$ in $N[G \restriction \lambda_n]$ and every $\bar{z} \in [\bigcup_{i \leq \omega} I_i]^{<\omega}$ in the domain of f, $f(\bar{z}) < \gamma_m$.

Suppose otherwise, and find a counterexample of such $m, n, t, f, \bar{z}$, minimizing first m and then n. Write $\bar{z} = (\bar{z}^0, \ldots, \bar{z}^n)$, with $\bar{z}^i \in I_i^{<\omega}$. Using condition (3) of the construction of the system and introducing more variables to f if necessary, choose $k < \omega$ so that $p_{n,I_n} \leq \sigma_n^k(\bar{z}^n)$ and the value of $f(\bar{z}^0, \ldots, \bar{z}^n)$ is forced over $V[H_n]$ by $\sigma_n^k(\bar{z}^n)$ of the system. If $m \geq n$, then the value of $f(\bar{z}^0, \ldots, \bar{z}^n)$ forced by $\sigma_n^k(\bar{z}^n)$ must be below γ_n, since γ_n is a member of C_n. Therefore $m < n$.

Now define $f' : [\bigcup_{i<n} X_i]^{<\omega} \times [X_n]^{<\omega} \to \lambda_m$ so that $f'(\bar{x}^0, \ldots, \bar{x}^n)$ is the value of $f(\bar{x}^0, \ldots, \bar{x}^n)$ forced over $V[H_n]$ by $\sigma_n^k(\bar{x}^n)$, if σ_n^k is defined on $\bar{x}^n$ and the value it forces is less than λ_m, and 0 otherwise. For $\bar{x}^0, \ldots, \bar{x}^{n-1}$, there are $Y \in U_n$ and $\delta < \lambda_m$ so that $f'(\bar{x}^0, \ldots, \bar{x}^{n-1}, \bar{w}) = \delta$ for all increasing $\bar{w} \subseteq Y$ of the right length. By intersecting such Y for all possible $\bar{x}^0, \ldots, \bar{x}^{n-1}$, we can take Y in $N[H_n]$ independent of the choice of $\bar{x}^0, \ldots, \bar{x}^{n-1}$. Let $h(\bar{x}^0, \ldots, \bar{x}^{n-1})$ be this δ. By elementarity, $h \in N[H_n]$ since $\sigma_n^k, U_n \in N[H_n]$. Since h has domain which is a subset of $[\bigcup_{i<n} X_n]^{<\omega}$, by λ_{n-1}^+-closure of $\mathbb{P} \restriction [\kappa_n, \lambda_n)$ over V, h belongs to $V[G \restriction \lambda_{n-1}]$ and maps into λ_m.

Take $\bar{x}^0 = \bar{z}^0, \ldots, \bar{x}^{n-1} = \bar{z}^{n-1}$, and let Y be as above. Since $Y \in N[H_n]$, $\bar{z}^n \subseteq Y_n \subseteq Y$ and therefore

$$h(\bar{z}^0, \ldots, \bar{z}^{n-1}) = f(\bar{z}^0, \ldots, \bar{z}^n) \geq \gamma_m.$$

The existence of h contradicts the minimality of n. □

Theorem 5.7. *If there is a proper class of supercompact cardinals, then there is a class forcing extension so that every increasing ω-sequence of regular cardinals has a mutually stationary sequence on cofinality ω_1 which is not tightly stationary.*

Proof. Force with the poset $\mathbb{P}$ from the previous theorem to obtain $V[G]$, and then force with $\mathrm{Add}(\omega, \omega_1)$. Let K be generic for $\mathrm{Add}(\omega, \omega_1)$ over $V[G]$. Let $\langle \lambda_i : i < \omega \rangle$ be an increasing ω-sequence of regular cardinals greater than ω_1 in $V[G * K]$, and $\lambda = \sup_i \lambda_i$. Let $\langle \lambda_{i_n} : n < \omega \rangle$ be a subsequence of $\langle \lambda_i : i < \omega \rangle$ so that for each n, there are at least three cardinals of $V[G * K]$ between λ_{i_n} and $\lambda_{i_{n+1}}$. Note that since $\mathbb{P}$ is ω_2-closed, the poset $\mathrm{Add}(\omega, \omega_1)$ we use is actually a member of the ground model V.

As before, there is a sequence of V-supercompact cardinals $\langle \kappa_n : n < \omega \rangle$ so that $\lambda_{i_{n-1}} \leq \kappa_n' < \kappa_n < \lambda_{i_n}$ for all $0 < n < \omega$, where κ_n' is the predecessor of κ_n in the sequence of supercompact cardinals. Let $\mathbb{P}^{(n)} = \mathbb{Q}_{\kappa_n'}$ and $\mathbb{R}^{(n)} = \mathbb{P} \restriction \kappa_n' \times \mathbb{P} \setminus \kappa_n$ be the quotient of $\mathbb{P}$ by $\mathbb{P}^{(n)}$, with corresponding generics g_n and H_n, respectively. In the extension by $\mathbb{R}^{(n)}$, $\mathsf{MA}(\aleph_1)$ holds and κ_n remains supercompact.

Using Theorem 3.5, let $\langle S_{i_n} : n < \omega \rangle$, $S_{i_n} \subseteq \lambda_{i_n} \cap \mathrm{Cof}(\omega_1)$ for each n, be a stationary sequence in $V[G * K]$ that is not tightly stationary. In $V[G]$, there is a stationary $S_n' \subseteq S_{i_n}$ since $\mathrm{Add}(\omega, \omega_1)$ has size $< \lambda_n$.

Let $F : [\lambda]^{<\omega} \to \lambda$ be a function in $V[G * K]$. Because $\mathrm{Add}(\omega, \omega_1)$ is c.c.c., there is $F' : \omega \times [\lambda]^{<\omega} \to \lambda$ in $V[G]$ so that for any $\bar{x} \subseteq \lambda$, there is $j < \omega$ so that $F'(j, \bar{x}) = F(\bar{x})$. Work as in Theorem 5.3, using F' instead of F in the structure $\mathcal{A}$ and using λ_{i_n} and S_{i_n}'. Note that although the sequences $\langle \lambda_{i_n} : n < \omega \rangle$ and

$\langle S'_n : n < \omega\rangle$ are not necessarily in $V[G]$, each of their finite initial segments are. This, together with the fact that $\{\lambda_{i_n} : n < \omega\}$ is contained in a countable set in V, suffices in the proof of Theorem 5.3. The proof produces a structure M closed under F', and hence under F, with $\sup(M \cap \lambda_{i_n})$ in S'_n and hence in S_{i_n}. We conclude that $\langle S_{i_n} : n < \omega\rangle$ is mutually stationary in $V[G * K]$.

Trivially extend $\langle S_{i_n} : n < \omega\rangle$ to a stationary sequence $\langle S_i : i < \omega\rangle$ on $\langle \lambda_i : i < \omega\rangle$ so that $S_i = \lambda_i \cap \mathrm{Cof}(\omega_1)$ if $\lambda_i \notin \{\lambda'_{i_n} : n < \omega\}$. The sequence $\langle S_i : i < \omega\rangle$ is mutually stationary but not tightly stationary. □

6. Indestructibility under further Prikry forcing

As discussed in Section 4, the only plausible strategies to force positive answers to our questions seem to involve Prikry-type forcing. The goal of this section is to show that the property of the model of the previous section that all mutually stationary sequences are tightly stationary is indestructible under Prikry-type forcing.

We first check that this model can have the measurable cardinals necessary to support Prikry-type forcings.

Lemma 6.1. *Suppose that κ is a limit point of the sequence of supercompact cardinals $\langle \mu_\xi : \xi \in \mathsf{ON}\rangle$ and κ is measurable in B with $2^\kappa = \kappa^+$. Then κ remains measurable in $V[G]$, where G is generic for $\mathbb{P}$ of Theorem 5.3, and in $V[G * K]$, where K is generic for* $\mathrm{Add}(\omega, \omega_1)$ *over $V[G]$.*

Proof. That the addition of K preserves measurability follows from the Levy–Solovay Theorem. Forcing with the κ^+-distributive poset $\mathbb{P}\restriction[\kappa, \infty)$ preserves the fact that κ is measurable. Let H be generic for $\mathbb{P}\restriction[\kappa, \infty)$. It remains to check that forcing with $\mathbb{P}\restriction\kappa$ over $V[H]$ preserves measurability. The argument is a bit more involved than usual because we are working with products rather than iterations.

In $V[H]$, let $j : V[H] \to M$ be the ultrapower by a normal ultrafilter on κ and let $G\restriction\kappa$ be generic for $\mathbb{P}\restriction\kappa$ over $V[H]$. Our aim is to lift j to $V[G]$, where $G = G\restriction\kappa \times H$. In order to do this, we need to find an M-generic filter in $V[G]$ for $j(\mathbb{P}\restriction\kappa)$. We can factor $j(\mathbb{P}\restriction\kappa) = \mathbb{P}\restriction\kappa \times \mathbb{R}$, where $\mathbb{R} = j(\mathbb{P})\restriction[\kappa, j(\kappa))$.

First, $G\restriction\kappa$ is M-generic for $\mathbb{P}\restriction\kappa$. Now in $V[G]$ we construct an $M[G\restriction\kappa]$-generic for $\mathbb{R}$. Let $\langle \dot{D}_\alpha : \alpha < \kappa^+\rangle$ be an enumeration of names for all of the open dense subsets of $\mathbb{R}$ which are in $M[G\restriction\kappa]$, with each name $\dot{D}_\alpha \in M$. There are only κ^+ many of them since $\mathbb{R}$ has size $j(\kappa)$, and in $V[H]$

$$|P^M(j(\kappa))| = |j(2^\kappa)| = |j(\kappa^+)| = (\kappa^+)^\kappa = \kappa^+.$$

Let $\langle p_i : i < \kappa\rangle$ be an enumeration of $\mathbb{P}\restriction\kappa$. The poset $\mathbb{R}$ is κ^+-closed in M. Using this closure, we inductively define $\langle q_{\alpha,i} : \alpha < \kappa^+, i < \kappa\rangle$ an array of conditions in $\mathbb{R}$ (running through the lexicographic order on the index).

Suppose we have completed the construction up to some index (α, i). Using the fact that $\dot{D}_\alpha$ is forced to be dense in $\mathbb{R}$, pick $q_{\alpha,i}$ to be some $q \in \mathbb{R}$ so that

(1) q is stronger than the previously defined conditions in the array;
(2) For some $p \le p_i$ in $\mathbb{P}\restriction\kappa$, $p \Vdash q \in \dot{D}_\alpha$.

Since ${}^\kappa M \subseteq M$, all proper initial segments of $\langle q_{\alpha,i} : \alpha < \kappa^+, i < \kappa\rangle$ belong to M, so the construction can proceed just using the κ^+-closure of $\mathbb{R}$ in M.

For each $\alpha < \kappa^+$, we have ensured that there is a dense set in M of $p \in \mathbb{P}\restriction\kappa$ which force $q_{\alpha,i} \in \dot{D}_\alpha$ for some $i < \kappa$, and $G\restriction\kappa$ meets each of these dense sets.

Therefore in $V[G]$, the set $\{q_{\alpha,i} : \alpha < \kappa^+, i < \kappa\}$ generates an $M[G\restriction\kappa]$-generic filter for $\mathbb{R}$, and so the embedding lifts. □

Now we define the class of forcing posets to which our main theorem will apply.

Definition 6.2. A poset (equipped with two partial orders) $(\mathbf{Pr}, \leq, \leq^*)$ is *κ-Prikry-like* if it satisfies the following conditions:

- $(\mathbf{Pr}, \leq, \leq^*)$ satisfies the Prikry property (i.e., condition (2) of 4.1).
- For every $\delta < \kappa$ and $p \in \mathbf{Pr}$, there is $q \leq p$ so that $(\mathbf{Pr}/q, \leq^*)$ is δ-closed.
- Write $p \sim q$ if and only if $p, q \in \mathbf{Pr}$ have a common $\leq^*$-extension. We require this to be an equivalence relation, each of which is centered in the $\leq^*$-order (has the property that any finite subset has a common $\leq^*$-lower bound).
- Suppose $f : \mathbf{Pr} \to \mathbf{Pr}$ is a selector for $\sim$, i.e., a function so that for all $p, q \in \mathbf{Pr}$,
  - $f(p) \sim p$,
  - if $p \sim q$ then $f(p) = f(q)$.

  Then for any $q \in \mathbf{Pr}$, there is $q^* \leq q$ so that q^* forces "if $p \leq q$ is in $G_{\mathbf{Pr}}$, then $f(p) \in G_{\mathbf{Pr}}$."

If $\mathbf{Pr}$ is κ-Prikry-like, let $\mathfrak{c}(\mathbf{Pr})$ denote the number of its $\sim$-equivalence classes.

Remark 6.3. The usual Prikry forcing with a normal ultrafilter U at κ satisfies the properties required of κ-Prikry-like forcing with $\mathfrak{c}(\mathbf{Pr}) = \kappa$. Any finitely many conditions with the same lower part have a common lower bound, and there are only κ many lower parts, so we can take $p \sim q$ if and only if p and q have the same stem. And given a function $f : \mathbf{Pr} \to \mathbf{Pr}$ as in the last condition, define A_s to be the upper part of $f(p)$ for any p with lower part s (this depends only on s). Intersecting $\Delta_\xi \bigcap_{s \subseteq \xi} A_s$ with the upper part of any q gives conditions q^* as required.

Remark 6.4. These conditions are also satisfied by the supercompact version of Prikry forcing and the diagonal versions of these forcings.

Theorem 6.5. *Suppose in $V[G * K]$ that $\mathbf{Pr}$ is a κ-Prikry-like forcing and that $(\mathbf{Pr}, \leq, \leq^*)$ is forcing isomorphic to some member of $V[G * K\restriction\gamma]$ for some $\gamma < \omega_1$.*

In the extension $V[G * K * G_{\mathbf{Pr}}]$, where $G_{\mathbf{Pr}}$ is $V[G * K]$-generic for $\mathbf{Pr}$, every increasing ω-sequence of regular cardinals with limit not in the interval $(\kappa, \mathfrak{c}(\mathbf{Pr}))$ has a mutually stationary sequence on cofinality ω_1 which is not tightly stationary.

Remark 6.6. If $\mathfrak{c}(\mathbf{Pr}) \leq \kappa^{+\omega}$, then the interval $(\kappa, \mathfrak{c}(\mathbf{Pr}))$ does not contain any singular cardinals and therefore the theorem applies to all increasing ω-sequence of regular cardinals in the extension.

Remark 6.7. The usual Prikry forcing defined over $V[G * K]$ is not a member of $V[G]$, but by the Levy–Solovay theorem U is generated by a $V[G]$-normal ultrafilter $\tilde{U} \in V[G]$, and the Prikry forcing in $V[G]$ defined from $\tilde{U}$ is dense in $\mathbf{Pr}$. In the case of diagonal Prikry forcing, the countable sequence of $V[G]$-ultrafilters appears in some initial segment of the extension by K.

Proof. Work below a condition where the number $\mathfrak{c}(\mathbf{Pr})$ of $\sim$-equivalence classes is forced.

The outline of the argument is as in Theorem 5.3: given an increasing sequence of regular cardinals of $V[G * K * G_{\mathbf{Pr}}]$, using the assumption that $\mathbf{Pr}$ is a member of $V[G * K\restriction\gamma]$ for some $\gamma < \omega_1$, the argument of Theorem 3.5 gives a sequence $\langle S_n : n < \omega\rangle$ of stationary subsets on cofinality ω_1 which is not tightly stationary. Our plan is to find a subsequence of the λ_n for which the restriction of $\langle S_n : n < \omega\rangle$ is mutually stationary—we will do this using arguments parallel to those in the proof of Theorem 5.7, even though the sequence is not in $V[G * G_{\mathbf{Pr}}]$. This will prove the theorem since the subsequence can be extended to a mutually stationary sequence which is not tightly stationary.

As before, it is enough to show that there is a mutually stationary but not tightly stationary sequence on each sequence of regular cardinals $\langle\lambda_n : n < \omega\rangle$ so that there are at least three $V[G]$ cardinals between λ_n and λ_{n+1} for each n. Let $\langle\kappa_n : n < \omega\rangle$ be cardinals in SC with $\lambda_{n-1} \leq \kappa'_n < \kappa_n < \lambda_n$, where κ'_n is the predecessor of κ_n in the sequence of supercompact cardinals. Let $F : [\kappa]^{<\omega} \to \kappa$ be a function in $V[G * K * G_{\mathbf{Pr}}]$. We will find $W \subseteq \sup_n \lambda_n$ in $V[G * K * G_{\mathbf{Pr}}]$ so that $\sup(F``[W]^{<\omega} \cap \lambda_n) \leq \sup(W \cap \lambda_n) \in S_n$ for every $n < \omega$.

For each $n < \omega$, let $\mathbb{R}^{(n)} = \mathbb{P}\restriction\kappa'_n \times \mathbb{P} \setminus \kappa_n$ and $\mathbb{P}^{(n)} = \mathrm{Col}^V((\kappa'_n)^+, <\kappa_n)$. Let g_n and H_n be the generics obtained from G for the posets $\mathbb{P}^{(n)}, \mathbb{R}^{(n)}$, respectively.

Case 1: $\sup_n \lambda_n < \kappa$. Forcing with $\mathbf{Pr}$ does not add bounded subsets of κ so we do not change the situation from that of Theorem 5.3.

Case 2: $\sup_n \lambda_n = \kappa$. By indestructibility, κ_n is supercompact in $V[H_n]$, so there is a normal fine ultrafilter U_n on $[\lambda_n]^{<\kappa_n}$ in $V[H_n]$.

There are several differences here from Theorem 5.3. Firstly, we cannot take a submodel N as in that proof from V, since for example neither the sequence $\langle\lambda_n : n < \omega\rangle$ nor any of its subsequences are even contained in a countable set in V. We get around this by using a different model N_n for each stage of the construction there. We must ensure that N_n "end-extends" N_{n-1} so that in the final step of the proof where we choose a function $f \in N_n$ with domain $[\lambda_n]^{<\omega}$ which maps above the supremum of W in some λ_m, the function h we construct from f is in fact a member of N_{n-1}, thereby contradicting the minimality of n.

The proof of Theorem 5.3 runs in certain intermediate extensions of $V[G]$ where we have Martin's Axiom and closure of the collapse posets $\mathbb{Q}_\alpha$. Here, we must also deal with K, the generic for $\mathrm{Add}(\omega, \omega_1)$ over $V[G]$. The influence of forcing with K is harder to ignore than in the proof of Theorem 5.7, since it is needed to make sense of $\mathbf{Pr}$.

For each n, the restriction $F \cap ([\lambda_n]^{<\omega} \times \lambda_n)$ is a member of $V[G * K]$, so let $\dot{F}_n$ be an $\mathrm{Add}(\omega, \omega_1)$ name for it which is forced to have range bounded by λ_n. In $V[G]$, define $\tilde{F}_n : \omega \times [\lambda_n]^{<\omega} \to \lambda_n$ so that $\tilde{F}_n(i, \bar{x})$ enumerates the possible values of $\dot{F}_n(\bar{x})$ forced by members of $\mathrm{Add}(\omega, \omega_1)$ (there are only countably many such values by the c.c.c.). The point is that any set closed under $\tilde{F}_n$ will also be closed under $F \cap ([\lambda_n]^{<\omega} \times \lambda_n)$.

Work in $V[G * K * G_{\mathbf{Pr}}]$ and let θ be a regular cardinal large enough so that $\mathbf{Pr} \in H(\theta)$. For each $n < \omega$ and $p \in \mathbf{Pr}$, let

$$N_n^p = \mathrm{Sk}^{H(\theta)^{V[G]}}(\in, \lhd, G, \dot{\mathbf{Pr}}, \dot{p}, \langle\dot{\lambda}_i : i < \omega\rangle, \langle\dot{p}_i, \kappa_i, \lambda_i, U_i, \tilde{F}_i : i \leq n\rangle),$$

where

- $\lhd$ is a predicate for a well-ordering on $H(\theta)^{V[G]}$,
- $\langle\dot{\lambda}_i :< \omega\rangle$ is a fixed $\mathrm{Add}(\omega, \omega_1) \times \mathbf{Pr}$-name for the sequence,
- $\dot{p}_i$ is an $\mathrm{Add}(\omega, \omega_1)$-name in $V[G]$ for the $\lhd$-least $p_i \leq p$ in $G_{\mathbf{Pr}}$ so that $(\mathbf{Pr}/p_i, \leq^*)$ is $|V[G]_{\lambda_i+6}|^+$-closed and forces a value for λ_i.

By passing to a dense set, we may assume that $(\mathbf{Pr}, \leq^*)$ is ω_1-closed. For $n \leq m < \omega$ and t a Skolem term in the language of N^p_m, define a function $f^{m,t}_n : \mathbf{Pr} \to \mathbf{Pr}$ by setting $f^{m,t}_n(s)$ to be the $\lhd$-least $r \sim s$ that forces a value for λ_n and $t \cap V[G]_{\lambda_n+6}$, if such exists, and just the $\lhd$-least $r \sim s$ otherwise. Note that these functions are definable over $(H(\theta)[K]; \in, \triangle, G, \mathbf{Pr}, \langle \dot{\lambda}_i : i < \omega\rangle)$. So each of the functions belongs to each of $N^p_m[K]$. In particular each of the structures is closed under each of the functions. Using the closure of $\leq^*$ and centeredness, the set of all conditions of the form $f^{m,t}_n(s)$ has a lower bound, $f(s)$. By the definition of Prikry-like forcing, there is $p^* \leq p$ that forces $s \in G_{\mathbf{Pr}} \to f(s) \in G_{\mathbf{Pr}}$.

Claim 6.8. *For $n \leq m$, the condition p^* forces that*

$$N^p_m \cap V[G]_{\lambda_n+6} = N^p_n \cap V[G]_{\lambda_n+6}.$$

Proof. Clearly $N^p_n \subseteq N^p_m$, so it remains to show that

$$N^p_m \cap V[G]_{\lambda_n+6} \subseteq N^p_n \cap V[G]_{\lambda_n+6}.$$

Suppose t is a term in the language of N^p_m. We will show that $p^* \Vdash t \cap V[G]_{\lambda_n+6} \in N^p_n$.

Since $\mathbf{Pr}$ does not add sets of von Neumann rank below κ,

$$N^p_m \cap V[G]_{\lambda_n+6} \in V[G * K].$$

Using the Prikry property, for any $r \leq p$ so that $(\mathbf{Pr}/r, \leq^*)$ is $|V[G]_{\lambda_n+6}|^+$-closed and forces a value for λ_n, there is a $\leq^*$-extension of r that forces a value for $t \cap V[G]_{\lambda_n+6}$. By definition, $p_n \in N^p_n[K]$ satisfies these properties, and therefore $f^{m,t}_n(p_n) \in N^p_n[K]$ forces a value for $t \cap V[G]_{\lambda_n+6}$.

If $p^* \in G_{\mathbf{Pr}}$, then $p_n \in G_{\mathbf{Pr}} \to f(p_n) \in G_{\mathbf{Pr}} \to f^{m,t}_n(p_n) \in G_{\mathbf{Pr}}$, so the value of $N^p_m \cap V[G]_{\lambda_n+6}$ forced by $f^{m,t}_n(p_n)$ would be the correct one, and as $f^{m,t}_n(p_n) \in N^p_n[K]$ we also have $t \cap V[G]_{\lambda_n+6} \in N^p_n[K]$. Since $\mathrm{Add}(\omega, \omega_1)$ is c.c.c., $t \cap V[G]_{\lambda_n+6} \in N^p_n$. As t was arbitrary we can conclude that $N^p_m \cap V[G]_{\lambda_n+6} \subseteq N^p_n$. □

By genericity, there is some $p \in \mathbf{Pr}$ so that $p^* \in G_{\mathbf{Pr}}$. Define $N_n = N^p_n \cap V_{\lambda_n+6}$. Note that $N_n \in V$ and $\langle \kappa_i, \lambda_i : i \leq n\rangle \in N_n$. The rest of the proof goes through as in Theorem 5.3, constructing the system at level n to decide values of functions in $N_n[G\restriction\lambda_n * K]$. We should be careful to ensure that the function h defined at the end of the proof is in $N_{n-1}[G\restriction\lambda_{n-1}]$, while the proof of Theorem 5.3 only showed that it was in $N[G\restriction\lambda_{n-1}]$. This holds because there must be a canonical $\mathrm{Add}(\omega, \omega_1) * \mathbf{Pr}$-name for h in $\tilde{N}$, and this must be in N and have rank at most $\lambda_{n-1} + 5$, and is therefore a member of N_{n-1}.

Case 3: $\sup_n \lambda_n > \mathfrak{c}(\mathbf{Pr})$. By ignoring an initial segment of the λ_n, we may assume that $\kappa_0 > \mathfrak{c}(\mathbf{Pr})$. In $V[G]$, suppose $\dot{\tau}$ is a $\mathrm{Add}(\omega, \omega_1) * \mathbf{Pr}$ name for an ordinal. There are at most $\aleph_1 \times \mathfrak{c}(\mathbf{Pr})$ many possibilities for the value of $\dot{\tau}[G_{\mathbf{Pr}}]$. Define $\tilde{F} : [\sup_n \lambda_n]^{<\omega} \to \sup_n \lambda_n$ so that $\tilde{F}(n, \bar{x})$ is the supremum of the possible values of $F(\bar{x})$ which are below λ_n. Then $\tilde{F} \in V[G]$, and for any $W \subseteq \sup_n \lambda_n$ and $n < \omega$, $\sup(F``[W]^{<\omega} \cap \lambda_n) \leq \sup(\tilde{F}``[W]^{<\omega} \cap \lambda_n)$. For each n, there is a λ_n-supercompactness measure U_n for κ_n in $V[H_n]$.

Let $\langle T_\alpha : \alpha < \mathfrak{c}(\mathbf{Pr})\rangle$ be an enumeration of the centered pieces of $\mathbf{Pr}$ in $V[G * K]$. Now argue similarly as in Theorem 5.3, using the same notation as in that proof but working with $\tilde{F}$ instead of F, and at stage n, replacing N (a countable Skolem hull of all relevant objects in V) with N_n, the Skolem hull of N together with some $\alpha_n < \mathfrak{c}(\mathbf{Pr})$ so that in $V[G*]$ there is $s \in T_{\alpha_n} \cap G_{\mathbf{Pr}}$ which forces values for

$\langle \kappa_i, \lambda_i, U_i : i \leq n\rangle$. Note that these values depend only on α_n, since the members of T_{α_n} are pairwise compatible. Since $\mathfrak{c}(\mathbf{Pr}) < \kappa_n$ there is a condition in $N \cap (H_n * K)$ which forces a value for the function that takes α_n to $\langle \kappa_i, \lambda_i, U_i : i \leq n\rangle$. Using the c.c.c., it follows that $\langle \kappa_i, \lambda_i, U_i : i \leq n\rangle \in N_n[H_n]$.

As before, for each n let $Y_n = \bigcap U_n \cap N_n[H_n]$ and construct a system $\langle \sigma_n^k : k < \omega\rangle$ with domain $Y_n \in U_n$ and stem $p_n \in g_n$. Using Martin's Axiom, let $p_{n,I}$ be defined as before.

Let C_n be the club in λ_n of closure points of the function taking an ordinal ξ to the supremum of values forced by the system applied to $[\xi]^{<\kappa_n}$. A version of Claim 5.6 holds, but the proof must be modified as we cannot find in $V[H_n * K]$ a stationary subset S'_n of S_n.

Claim 6.9. *For each $n < \omega$, there is $I_n \subseteq Y_n$ which is $<$-increasing of order-type ω_1 so that $p_{n,I_n} \in g_n$ with $\gamma_n := \sup(\bigcup I_n) \in C_n \cap S_n$.*

Proof of Claim 6.9. Work in $V[H_n * K]$. Consider the set of $(p, \dot{r}) \in \mathbb{P}^{(n)} * \mathbf{Pr}$ for which there is some $I \subseteq Y_n$ which is $<$-increasing of order-type ω_1 so that:

(1) $p \leq p_{n,I}$.
(2) $(p, \dot{r}) \Vdash \sup(\bigcup I) \in C_n \cap \dot{S}_n$.

We will show that this set is dense below $(p_n, 1_{\mathbf{Pr}})$, and therefore intersects $g_n * G_{\mathbf{Pr}}$.

Let $q \leq p_n$ in $\mathbb{P}^{(n)}$ and $\dot{s}$ be a $\mathbb{P}^{(n)}$-name for an element of $\mathbf{Pr}$. In $V[G]$, there is a stationary $S_n^* \subseteq \lambda_n$ so that for any $\gamma \in S_n^*$, there is some $s' \leq s$ with $s' \Vdash \gamma \in \dot{S}_n$. Since $|\mathbb{P}^{(n)}| < \lambda_n$, there are $q' \leq q$ and $S'_n \in V[H_n * K]$ so that S'_n is stationary and $q' \Vdash S'_n \subseteq \dot{S}_n^*$. In total, we have defined q' and S'_n so that for any $\gamma \in S'_n$, there is some $\dot{s}'$ so that $q' \Vdash \dot{s}' \leq \dot{s}$ and $(q', \dot{s}') \Vdash \gamma \in \dot{S}_n$.

The elements of $\mathbb{P}^{(n)}$ have domains of size $< \kappa_n$, so there is $\beta < \kappa_n$ with $\mathrm{dom}(q') = \mathrm{dom}(q' \restriction \beta)$. Pick $I \subseteq Y_n$ to be $<$-increasing of order-type ω_1 with $\min(I) \cap \kappa_n > \beta$ and $\sup(\bigcup I) \in S'_n$.

Choose $\dot{r}$ a name for a condition in $\mathbf{Pr}$ so that $q' \Vdash \dot{r} \leq \dot{s}$ and $(q', \dot{r}) \Vdash \sup(\bigcup I) \in \dot{S}_n$. Since p_n is the stem of the system, p_n extends the restriction to $\min(I) \cap \kappa_n$ of any member of $\mathbb{P}_{n,I}$. So $p_{n,I} \restriction \beta \subseteq p_{n,I} \restriction (\min(I) \cap \kappa_n) \geq p_n$, and therefore $p_{n,I}$ is compatible with q'. Choosing $p \leq p_{n,I}, q'$ gives the result. $\square$

Take $W = \bigcup_n I_n$ and $\gamma_n = \sup(W \cap \lambda_n)$ for each $n < \omega$.

For each n, the restriction of $\tilde{F}$ to $\bar{x} \in [\lambda_n]^{<\omega}$ with $\tilde{F}(\bar{x}) < \lambda_n$ is an element of $N_n[G \restriction \lambda_n]$. The image of W under $\tilde{F}$ is the union of its images under these functions. Thus it is enough to prove for each n and each $f \in N_n[G \restriction \lambda_n]$ that

$$\sup(f``[W \cap \lambda_n]^{<\omega} \cap \lambda_m) \leq \gamma_m$$

for each m. We can restrict attention to functions on $[\bigcup_{i \leq n} X_i]^{<\omega}$, since $W \cap \lambda_n = \bigcup_{i \leq n} I_i \subseteq \bigcup_{i \leq n} X_i$. We can further restrict to functions f mapping into λ_m. We then have to prove that for every $\bar{z} \in [\bigcup_{i \leq n} I_i]^{<\omega}$, $f(\bar{z}) < \gamma_m$.

Suppose otherwise, and fix a counterexample m, n, f, and $\bar{z} = (\bar{z}^0, \dots, \bar{z}^n)$ with $\bar{z}^i \in I_i^{<\omega}$, minimizing first m and then n. As in Section 3, we have $m < n$ and the value of $f(\bar{z}^0, \dots, \bar{z}^n)$ is forced over $V[H_n]$ by some condition $\sigma_n^k(\bar{z}^n)$ of the system where $p_{n,I_n} \leq \sigma_n^k(\bar{z}^n)$.

Recall that σ_n^k belongs to the model $N_n[H_n]$, and its domain is a certain set Y^k. Define $f' : [\bigcup_{i<n} X_n]^{<\omega} \times Y_n \to \lambda_m$ so that for $\bar{x}^0, \dots, \bar{x}^n$ of the appropriate type, $f'(\bar{x}^0, \dots, \bar{x}^n)$ is the value of $f(\bar{x}^0, \dots, \bar{x}^n)$ forced over $V[H_n]$ by $\sigma_n^k(\bar{x}^n)$ (if such exists, and 0 otherwise). There is a $Y \in U_n \cap N[H_n]$ so that for every $\bar{x}^0, \dots, \bar{x}^{n-1}$ there is some $\delta < \lambda_m$ so that

$$f'(\bar{x}^0, \dots, \bar{x}^{n-1}, \bar{w}) = \delta$$

for all increasing $\bar{w} \subseteq Y$ of the appropriate length. Let $h(\bar{x}^0, \dots, \bar{x}^{n-1})$ be this fixed value.

Now $Y_n \subseteq Y$ and therefore

$$f(\bar{z}^0, \dots, \bar{z}^{n-1}, \bar{z}^n) = h(\bar{z}^0, \dots, \bar{z}^{n-1}).$$

Using the λ_{n-1}^+-closure of $\mathbb{P}^{(n)}$, we have $h \in N_n[G\restriction\lambda_{n-1}]$. Since $N_n[G\restriction\lambda_{n-1}]$ is the Skolem hull of $N_{n-1}[G\restriction\lambda_{n-1}]$ together with α_n, there is a function $h' : \mathbf{Pr} \times [\bigcup_{i \leq n} X_i]^{<\omega} \to \lambda_m$ in $N_{n-1}[G\restriction\lambda_{n-1} * K]$ so that $h'(\alpha_n, \bar{x}) = h(\bar{x})$ for all $\bar{x}$ in the domain of h. In $N_{n-1}[G\restriction\lambda_{n-1}]$, we can define $h''(\bar{x}) = \sup_{\alpha < \mathfrak{c}(\mathbf{Pr})} h'(\alpha, \bar{x})$. Since λ_m has cofinality larger than $\mathfrak{c}(\mathbf{Pr})$, $h''(\bar{z}^0, \dots, \bar{z}^{n-1}) < \lambda_m$ and

$$\begin{aligned} h''(\bar{z}^0, \dots, \bar{z}^{n-1}) &\geq h'(\alpha_n, \bar{z}^0, \dots, \bar{z}^{n-1}) \\ &= h(\bar{z}^0, \dots, \bar{z}^{n-1}) \\ &\geq f(\bar{z}^0, \dots, \bar{z}^{n-1}, \bar{z}^n) \\ &\geq \gamma_m. \end{aligned}$$

As $h'' \in N_{n-1}[G\restriction\lambda_{n-1}]$, this contradicts the minimality of n.

□

7. Getting stronger large cardinals using iterations

The model $V[G * K]$ of Section 6 has only limited large cardinals. In this section we will obtain another model, $V[G^* * K]$, with the same separation of mutual from tight stationarity, and the same indestructibility of this separation under Prikry-like forcing, but with G^* generic for an iteration rather than a product forcing. By well known standard facts this new model inherits large cardinals of arbitrarily large strength from V. In particular, any supercompact limit of supercompact cardinals in V remains supercompact in $V[G^* * K]$, and similarly for many stronger axioms.

Define an Easton support iteration $\mathbb{P}^*$ of class length, where the factor $\dot{\mathbb{Q}}^*_\alpha$ is the standard $\mathbb{P}\restriction\xi$-name for the poset $\mathrm{Col}(\mu_\xi^+, < \mu_{\xi+1})$ if $\alpha = \mu_\xi \in \mathsf{SC}$, and trivial otherwise.

We use the termspace forcing construction to relate the iteration $\mathbb{P}^*$ to the product $\mathbb{P}$ we have considered in the previous sections. Following [8], for $\mathbb{P}$ a forcing poset and $\mathbb{Q}$ a $\mathbb{P}$-name for a forcing poset, define $A(\mathbb{P}, \dot{Q})$ to be the poset whose elements are canonical $\mathbb{P}$-names for elements of $\dot{Q}$, ordered by $\dot{\sigma} \leq \dot{\tau}$ iff $\Vdash_{\mathbb{P}} \dot{\sigma} \leq_{\mathbb{Q}} \dot{\tau}$.

We collect some basic facts about termspace forcing that will be useful in what follows. See [8] for a reference.

Fact 7.1.
- If κ is a cardinal and $\dot{Q}$ is forced to be κ-closed, then $A(\mathbb{P}, \dot{Q})$ is κ-closed.
- Suppose G is generic for $\mathbb{P}$ and I is generic for $A(\mathbb{P}, \dot{Q})$. Then $H := \{\dot{\sigma}[G] : \dot{\sigma} \in I\}$ is generic for $\dot{Q}[G]$ over $V[G]$.
- If $\langle \mathbb{P}_i, \mathbb{Q}_i \rangle$ is a forcing iteration with supports in an ideal I, then the limit of the iteration can be completely embedded in the product of the posets $A(\mathbb{P}_i, \dot{Q}_i)$ taken with supports in I.

Lemma 7.2. *Suppose $\mu \in \mathsf{SC}$. Then $A(\mathbb{P}^*_\mu, \dot{Q}^*_\mu)$ completely embeds in $\mathbb{Q}_\mu$ with a μ^+-closed quotient.*

Proof. Let ν be the successor of μ in the sequence SC, so that ν is supercompact in V. The Levy collapse $\mathbb{Q}_\mu$ is forcing isomorphic to the $\leq \mu$-support product of the posets $\mathrm{Col}(\mu^+, \beta)$, $\beta < \nu$.

Suppose $\dot{q}$ is a $\mathbb{P}^*_\mu$-name for an element of $\dot{Q}^*_\mu$. Since $|\mathbb{P}^*_\mu| = \mu$, there is some $Z \subseteq \nu$ in V of size $\leq \mu$ so that it is forced by $\mathbb{P}^*_\mu$ that $\mathrm{dom}(\dot{q}) \subseteq \mu \times Z$. Therefore $A(\mathbb{P}^*_\mu, \dot{Q}^*_\mu)$ factors as the $\leq \mu$-support product of $A(\mathbb{P}^*_\mu, \dot{Q}^*_{\mu,\alpha})$, $\alpha < \nu$, where $\dot{Q}^*_{\mu,\alpha}$ is the name for the poset $\mathrm{Col}(\mu^+, \alpha)$.

For each $\alpha < \nu$, the poset $A(\mathbb{P}^*_\mu, \dot{Q}^*_{\mu,\alpha})$ has size $< \nu$ and is μ^+-closed, and therefore completely embeds in $\mathrm{Col}(\mu^+, \beta)$ with a μ^+-closed quotient for sufficiently large $\beta < \nu$. Let $h : \nu \to \nu$ be an increasing function so that $A(\mathbb{P}^*_\mu, \dot{Q}^*_{\mu,\alpha})$ completely embeds into $\mathrm{Col}(\mu^+, h(\alpha))$ for each $\alpha < \nu$. Putting these embeddings together in the $\leq \mu$-support product gives the complete embedding from $A(\mathbb{P}^*_\mu, \dot{Q}^*_\mu)$ to $\mathbb{Q}_\mu$. $\square$

For the next theorem, we will need some facts about the approachability ideal $I[\lambda]$. Recall that for an uncountable regular cardinal λ, S is in $I[\lambda]$ if $S \subseteq \lambda$ and there is a sequence $\langle d_i : i < \lambda \rangle$ of bounded subsets of λ and a club $E \subseteq \lambda$ so that for every $\alpha \in S \cap E$, there is $A_\alpha \subseteq \alpha$ so that $\mathrm{ot}(A_\alpha) = \mathrm{cf}(\alpha)$ and for every $\beta < \alpha$, there is some $i < \alpha$ so that $A_\alpha \cap \beta = d_i$. We state the well-known facts about $I[\lambda]$ that we will need (see [2] for a reference).

Fact 7.3. Let η, λ be regular cardinals with $\eta^+ < \lambda$.

- There is a set $S \in I[\lambda]$ so that $S \subseteq \mathrm{Cof}(\eta) \cap \lambda$ is stationary.
- If $S \subseteq \mathrm{Cof}(\eta) \cap \lambda$ is stationary and in $I[\lambda]$, then it remains stationary after forcing with an η^+-closed poset.

Theorem 7.4. *Let G^* be generic for $\mathbb{P}^*$ over V and K be generic for* $\mathrm{Add}(\omega, \omega_1)$ *over $V[G^*]$. Then in $V[G^* * K]$, every increasing ω-sequence of regular cardinals has a mutually stationary sequence on cofinality ω_1 which is not tightly stationary.*

Proof. Let $\langle \lambda_i : i < \omega \rangle$ be an increasing ω-sequence of regular cardinals greater than ω_1 in $V[G^* * K]$, and $\lambda = \sup_n \lambda_n$. Let $\langle \lambda_{i_n} : n < \omega \rangle$ be a subsequence of $\langle \lambda_i : i < \omega \rangle$ so that for each n, there are at least three cardinals of $V[G^* * K]$ between λ_{i_n} and $\lambda_{i_{n+1}}$. Using Theorem 3.5, let $\langle S_{i_n} : n < \omega \rangle$, $S_{i_n} \subseteq \lambda_{i_n} \cap \mathrm{Cof}(\omega_1)$ for each n, be a stationary sequence in $V[G^* * K]$ that is not tightly stationary. Additionally, the proof of Proposition 3.3 is flexible enough to choose $S_{i_n} \in I[\lambda_n] \cap V[G^*]$ for each $n < \omega$.

The iteration $\mathbb{P}^*$ completely embeds (by Fact 7.2) in the Easton support product of the posets $A(\mathbb{P}^*_\mu, \dot{Q}^*_\mu)$, $\mu < \lambda$, and therefore completely embeds in $\mathbb{P}$ by Lemma 7.2. Let G be generic for $\mathbb{P}$ extending the image of G^* under this embedding. The quotient of the complete embedding of $\mathbb{P}^*$ into $\mathbb{P}$ is ω_2-closed and S_{i_n} remains in $I[\lambda_n]$, so S_{i_n} is stationary in $V[G]$ by Fact 7.3, and therefore is stationary in $V[G * K]$. By Theorem 5.7, $\langle S_{i_n} : n < \omega \rangle$ is mutually stationary in $V[G * K]$. Each algebra $\mathcal{A}$ in $V[G^* * K]$ belongs to $V[G * K]$, and the set in $V[G * K]$ closed under $\mathcal{A}$ constructed by Theorem 5.7 to meet the sequence $\langle S_{i_n} : n < \omega \rangle$ is of size ω_1, so belongs to $V[G^* * K]$. Therefore $\langle S_{i_n} : n < \omega \rangle$ is mutually stationary in $V[G^* * K]$.

This sequence can be extended in a trivial way to $\langle S_n : n < \omega \rangle$ which is mutually but not tightly stationary. $\square$

Theorem 7.5. *Let* $(\mathbf{Pr}, \leq, \leq^*)$ *be a* κ*-Prikry-like forcing notion in* $V[G^* * K]$ *which is forcing-isomorphic to some member of* $V[G * K \restriction \gamma]$ *for some* $\gamma < \omega_1$.

In the extension $V[G^* * K * G_{\mathbf{Pr}}]$*, where* $G_{\mathbf{Pr}}$ *is* $V[G^* * K]$*-generic for* $\mathbf{Pr}$*, every increasing* ω*-sequence of regular cardinals with limit not in the interval* $(\kappa, \mathfrak{c}(\mathbf{Pr}))$ *has a mutually stationary sequence on cofinality* ω_1 *which is not tightly stationary.*

Proof. It is enough to show that there is a mutually stationary but not tightly stationary sequence on each sequence of regular cardinals $\langle \lambda_n : n < \omega \rangle$ having at least three $V[G^*]$ cardinals between λ_n and λ_{n+1} for each n. Let $\langle \kappa_n : n < \omega \rangle$ be formerly supercompact cardinals with $\lambda_{n-1} \leq \kappa'_n < \kappa_n < \lambda_n$, where κ'_n is the predecessor of κ_n in SC. In $V[G^* * K * \mathbf{Pr}]$, let $\langle S_n : n < \omega \rangle$ be a stationary sequence on $\langle \lambda_n : n < \omega \rangle$ which is not tightly stationary so that $S_n \in I[\lambda_n]$ concentrates on points of cofinality ω_1 for every n. Let $F : [\sup_n \lambda_n]^{<\omega} \to \sup_n \lambda_n$ be a function in $V[G^* * K * G_{\mathbf{Pr}}]$. We will find $W \subseteq \sup_n \lambda_n$ in $V[G^* * K * G_{\mathbf{Pr}}]$ so that $\sup(F``[W]^{<\omega} \cap \lambda_n) \leq \sup(W \cap \lambda_n) \in S_n$ for every $n < \omega$.

Case 1: $\sup_n \lambda_n < \kappa$. Forcing with $\mathbf{Pr}$ over $V[G^* * K]$ does not add bounded subsets of κ so we do not change the situation from that of Theorem 7.4.

Case 2: $\sup_n \lambda_n = \kappa$. By Lemma 7.2, we can force over $V[G^*]$ by some quotient poset to obtain $V[G]$, where G is generic for the product $\mathbb{P}$.

The argument is similar to those of Theorems 5.3 and 6.5. Let $\mathfrak{c}(\mathbf{Pr})$ be defined as in Theorem 6.5.

We define N_n similarly as in case 2 of the proof of Theorem 6.5. However, the poset $\mathbf{Pr}$ does not necessarily have the Prikry property over $V[G]$, so it may add bounded subsets of κ. Thus, take the Skolem hulls defining the N_n^p in $V[G^*]$ rather than $V[G]$, and construct the N_n in $V[G^*]$. By the closure of the quotient forcing, each model N_n can be extended to $N'_n \in V[G]$, a $V[G]$-Skolem hull of the relevant objects, with $N'_n \cap V[G^*] = N_n$ and therefore the crucial property that $N'_n \cap V[G]_{\lambda_n+6} = N'_m \cap \lambda_n + 6$ for all $n \leq m < \omega$. These N'_n contain the appropriate restrictions of F, since the N_n do, so the arguments concluding the proofs in Theorems 5.3 and 5.7 go through to show that in $V[G * K * G_{\mathbf{Pr}}]$ there is a $W \subseteq \kappa$ so that $\sup(F``W \cap \lambda_n) = \sup(W \cap \lambda_n) \in S_n$ for every n. Moreover, the proper initial segments of W are all in V.

We now show that there must exist a set with these properties in $V[G^* * K * G_{\mathbf{Pr}}]$. Consider the tree $\mathcal{T}$ of attempts to construct such a set, so that the nth level of $\mathcal{T}$ consists of $w \subseteq \lambda_n$ so that $\sup(F``w \cap \lambda_i) = \sup(w \cap \lambda_i) \in S_i$ for all $i \leq n$ (and the ordering on $\mathcal{T}$ is by end-extension). The initial segments of W give an infinite branch through $\mathcal{T}$ in $V[G * K * G_{\mathbf{Pr}}]$, so by the absoluteness of well-foundedness, there is in $V[G^* * K * G_{\mathbf{Pr}}]$ an infinite branch through $\mathcal{T}$ and hence a $W^* \subseteq \kappa$ so that $\sup(F``W^* \cap \lambda_n) = \sup(W^* \cap \lambda_n) \in S_n$ for every n.

Case 3: $\sup_n \lambda_n > \kappa$. The argument of case 3 in the proof of Theorem 6.5 adapts to show that in $V[G * K * G_{\mathbf{Pr}}]$ there is a $W \subseteq \sup_n \lambda_n$ so that $\sup(F``W \cap \lambda_n) = \sup(W \cap \lambda_n) \in S_n$ for every n. This is because the argument only used the centeredness of $\mathbf{Pr}$, which persists from $V[G^* * K]$ to $V[G * K]$. The result follows from the absoluteness argument of the previous case. □

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