

CARDINAL CHARACTERISTICS OF THE CONTINUUM AND PARTITIONS

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ABSTRACT. We prove that for regular cardinals κ , combinations of the stick principle at κ and certain cardinal characteristics at κ being κ^+ causes partition relations such as $\kappa^+ \rightarrow (\kappa^+, (\kappa : 2))^2$ and $(\kappa^+)^2 \rightarrow (\kappa^+ \kappa, 4)^2$ to fail. Polarised partition relations are also considered, and the results are used to answer several problems posed by Garti, Larson and Shelah.

1. INTRODUCTION

In this paper we consider the effect of cardinal characteristics of the continuum on partition relations. Connections between these two areas have been investigated before, a recent contribution is [014RT]. For a general overview of cardinal characteristics, cf. [010Bl, 012Ha], for partition relations cf. [977Wi, 984E, 010HL].

For an ordered set X and an ordinal α , let $[X]^\alpha$ denote the subsets of X of order-type α . In Rado's notation, cf. [956ER], given ordinal parameters,

$$(1) \quad \alpha \longrightarrow (\beta_i : i < m)_m^k$$

is the statement that for any $f : [\alpha]^k \rightarrow m$, there are $i < m$ and $X \in [\alpha]^{\beta_i}$ for which $f \upharpoonright [X]^k$ is constant with value i (we say that X is *homogeneous* for i).

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We will also consider polarised partition relations. There,

$$(2) \quad \binom{\alpha}{\beta} \longrightarrow \binom{\gamma_i}{\delta_i} : i < m \Bigg|_m^{1,1}$$

means that for every $f : \alpha \times \beta \rightarrow m$, there are $i < m$, $X \in [\alpha]^{\gamma_i}$, and $Y \in [\beta]^{\delta_i}$ so that $f \upharpoonright [X \times Y]$ is constant with value i .

The notation is flexible in several ways. Replacing \longrightarrow with $\not\rightarrow$ gives the negation of the original statement. If there is β so that $\beta_i = \beta$ for all $i < m$, then we will drop the indexing on the β_i . In this case, if the number of colours is 2, we often omit the lower subscript. Replacing the parentheses on the right-hand side by square brackets weakens the statement so that we do not require X to be homogeneous but just so that $f[X] \neq m$ (i.e., X misses one colour, instead of missing all colours but one, cf. [965E, Definition 18.1]).

Finally, we can replace β_i by $(\beta_i : \gamma_i)$. This notation means that instead of asking for a homogeneous set for i of order-type β_i , we ask for X of order-type β_i and Y of order-type γ_i so that $x < y$ and $f(x, y) = i$ for every $x \in X$ and $y \in Y$. This is weaker than requiring a homogeneous set for i of order-type $\beta_i + \gamma_i$.

In early work in this area, many negative partition relations involving ordinals $\leq \omega_1$ were shown to follow from the assumption of CH. Subsequently, this assumption was reduced in many cases to $\mathfrak{cc} = \aleph_1$, where \mathfrak{cc} is a cardinal characteristic. This gives a way of calibrating more precisely how much of CH is actually needed to prove a negative partition relation.

For a cardinal κ , the following cardinal characteristics will be used in this paper:

- (1) The *unbounding number* \mathfrak{b}_κ is the least cardinality of a family $A \subseteq {}^\kappa\kappa$ so that for any $f \in {}^\kappa\kappa$ there is a $g \in A$ so that $|\{\alpha < \kappa \mid f(\alpha) < g(\alpha)\}| = \kappa$.
- (2) The *dominating number* \mathfrak{d}_κ is the least cardinality of a family $A \subseteq {}^\kappa\kappa$ so that for any $f \in {}^\kappa\kappa$ there is a $g \in A$ so that $|\{\alpha < \kappa \mid g(\alpha) < f(\alpha)\}| < \kappa$.
- (3) The *reaping number* \mathfrak{r}_κ (sometimes called “refining number” or “unsplitting number”) is the least cardinality of a family $A \subseteq [\kappa]^\kappa$ such that for any $X \in [\kappa]^\kappa$ there is $Y \in A$ so that $\min(|Y \cap X|, |Y \setminus X|) < \kappa$.
- (4) The *splitting number* \mathfrak{s}_κ is the least cardinality of a family $A \subseteq [\kappa]^\kappa$ such that for any $X \in [\kappa]^\kappa$ there is $Y \in A$ so that $|X \cap Y| = |X \setminus Y| = \kappa$.

(5) $\text{cof}(\mathcal{M})$ is the least cardinality of a family \mathcal{F} of meagre subsets of ${}^\omega 2$ so that for any meagre subset A of ${}^\omega 2$ there is $B \in \mathcal{F}$ with $A \subseteq B$.

For every cardinal characteristic $\mathfrak{cc} \in \{\mathfrak{b}, \mathfrak{d}, \mathfrak{r}, \mathfrak{s}\}$ the meaning of \mathfrak{cc} is \mathfrak{cc}_ω . It can be proven by ZFC that $\mathfrak{b}_\kappa \leq \mathfrak{d}_\kappa$, $\mathfrak{s}_\kappa \leq \mathfrak{d}_\kappa$, and $\mathfrak{b}_\kappa \leq \mathfrak{r}_\kappa$ hold for indecomposable κ (cf. Blass's chapter [010B1] in the Handbook of Set Theory for $\kappa = \omega$ which easily generalises). These characteristics each have value at most 2^κ , moreover, if κ is both regular and uncountable, then $\mathfrak{s}_\kappa \leq \mathfrak{b}_\kappa$, cf. [017RS].

We also consider another class of cardinal characteristics, sticks, we follow the terminology established in [006Br].

(3) $\mathfrak{I}(\kappa, \lambda) := \min \{|X| \mid X \subseteq [\lambda]^\kappa \wedge \forall y \in [\lambda]^\lambda \exists x \in X : x \subseteq y\}$ for both κ and λ cardinals with $\kappa \leq \lambda$.

We will write $\mathfrak{I}(\kappa)$ for $\mathfrak{I}(\kappa, \kappa^+)$. \mathfrak{I} simply stands for $\mathfrak{I}(\aleph_0)$, cf. [997F] and sometimes for $\mathfrak{I}(\aleph_0) = \aleph_1$, cf. [978B]. Note that $\mathfrak{I}(\kappa, \lambda) = \mu$ is denoted by $A(\lambda, \mu, \kappa, \kappa^+)$ in [976Ba].

$\mathfrak{I}(\kappa)$ is not typically considered a cardinal characteristic of 2^κ since it involves the combinatorics at κ^+ , but as such characteristics it also takes a value greater than κ and at most 2^κ . Truss [983Tr] showed that if $\mathfrak{I} = \aleph_1$, then either the covering number of the meagre ideal or the covering number of the Lebesgue-null ideal is also \aleph_1 . Brendle [006Br] further considered the relationship between \mathfrak{I} and other cardinal characteristics.

Now we give an overview of our main results and provide some context.

By the Dushnik–Miller Theorem—cf. [941DM, Theorem 5.23]—we have $\omega_1 \rightarrow (\omega_1, \omega + 1)^2$. From CH, Hajnal [960Ha] proved that $\omega_1 \not\rightarrow (\omega_1, \omega + 2)^2$. A recent paper of Raghavan and Todorćević [016RT] shows that if there is a Suslin tree, then $\omega_1 \not\rightarrow (\omega_1, \omega + 2)^2$. We can arrive at the same conclusion from the hypothesis $\mathfrak{I} = \aleph_1$ (Theorem 3.1).

Starting from GCH, Erdős and Hajnal [971EH] proved $\kappa^+ \kappa \not\rightarrow (\kappa^+ \kappa, 3)^2$ for all cardinals κ and Hajnal [971Ha] proved $\omega_1^2 \not\rightarrow (\omega_1^2, 3)^2$ from CH. The hypothesis of CH was reduced in several different ways.

Fact 1.1. (1) (Takahashi [987Ta]) If $\mathfrak{I} = \aleph_1$, then $\omega_1^2 \not\rightarrow (\omega_1^2, 3)^2$.

(2) (Todorćević [989To]) If $\mathfrak{b} = \aleph_1$, then $\omega_1 \not\rightarrow (\omega_1, \omega + 2)^2$.

(3) (Larson [998La]) If κ is regular and $\mathfrak{d}_\kappa = \kappa^+$, then $\kappa^+ \kappa \not\rightarrow (\kappa^+ \kappa, 3)^2$ and $(\kappa^+)^2 \not\rightarrow ((\kappa^+)^2, 3)^2$.

(4) (Lambie-Hanson and Weinert [017LW]) If $\mathfrak{b}_\kappa = \mathfrak{I}(\kappa) = \kappa^+$, then $\kappa^+ \kappa \not\rightarrow (\kappa^+ \kappa, 3)^2$.

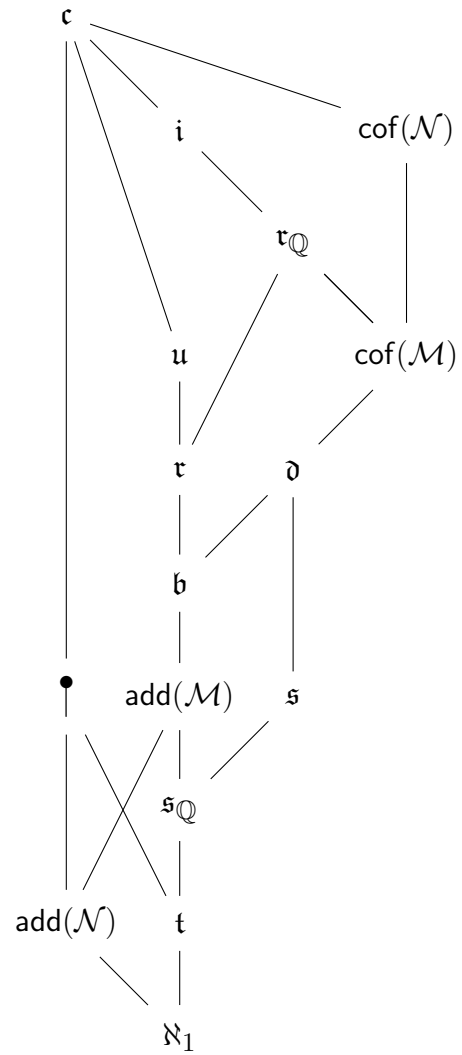


FIGURE 1. A diagram

Baumgartner and Hajnal [987BH] proved that $2^\kappa = \kappa^+ = \lambda$ for a regular κ implies that $\lambda^2 \not\rightarrow (\lambda\kappa, 4)^2$. In this paper, we reduce the hypothesis used to $\mathfrak{d}_\kappa = \kappa^+ = \lambda$ for a regular κ (Theorem 2.9), answering a question of Larson, and to $\mathfrak{b}_\kappa = \mathfrak{b}(\kappa) = \kappa^+ = \lambda$ for a regular κ (Theorem 4.3). For $\kappa = \omega$ it is known that for this result as well as for Fact 1.1 (3) & (4) a hypothesis is needed as Baumgartner proved in [989Ba] that $\text{ZFC} + \text{MA}_{\aleph_1}$ implies that $\omega_1\omega \rightarrow (\omega_1\omega, n)^2$ for all natural numbers n . For 1.1 (2) this was shown by Todorćević in [983To] and for Fact 1.1 (1) this is as of now unknown.

For polarised partition relations, Garti and Shelah [014GS] proved the following:

- Fact 1.2.** (1) If $\kappa < \mathfrak{s}$ and $\text{cf}(\kappa) > \omega$, then $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}_2^{1,1}$.
 (2) If $\mathfrak{r} < \kappa \leq \mathfrak{c}$ and $\text{cf}(\kappa) > \mathfrak{r}$, then $\binom{\kappa}{\omega} \rightarrow \binom{\kappa}{\omega}_2^{1,1}$.
 (3) If $\mathfrak{r} = \mathfrak{c}$ and $\kappa \in [\text{cf}(\mathfrak{c}), \mathfrak{c}]$, then $\binom{\kappa}{\omega} \not\rightarrow \binom{\kappa}{\omega}_2^{1,1}$.

In Theorem 2.7 we are able to prove that $\mathfrak{d} = \aleph_1$ implies $(\omega_1) \not\rightarrow (\omega_1)_2^{1,1}$ (and in fact something a bit stronger, increasing the number of colours and obtaining a negative square-bracket relation). As corollaries, we answer several questions from [016GS].

In the last section, we relate Luzin sets to partition relations, showing that the existence of a Luzin set suffices for an example of Shelah of a function witnessing $\omega_1 \not\rightarrow [\omega_1]_\omega^2$ with no triangle having three different colours.

2. PARTITION RELATIONS AND THE DOMINATING NUMBER

We recall the following definition.

Definition 2.1. Let α and β both be ordinals. A sequence of functions $\langle f_\xi \mid \xi < \rho \rangle$ in ${}^\alpha\beta$ is called a *scale* if the order of eventual domination (i.e. $f_\gamma < f_\delta$ if there is a $\zeta < \alpha$ such that for all $\xi \in \rho \setminus \zeta$ we have $f_\gamma(\xi) < f_\delta(\xi)$) is a well-order. It is called *unbounded* if for all $f \in {}^\alpha\beta$ there is a $\xi < \rho$ such that $f < f_\xi$.

We will be mostly interested in the case where α is a cardinal and $\alpha = \beta$. Note that what we call a scale is called a *strict scale* in [979Co]. One can inductively define a scale of length \mathfrak{b} . If $\mathfrak{b} = \mathfrak{d}$ then one can define an unbounded scale of length \mathfrak{b} .

Theorem 2.2. *Suppose that $\mathfrak{b} = \mathfrak{d}$. Then*

$$\left(\begin{array}{c} \mathfrak{d} \\ \omega \end{array} \right) \not\rightarrow \left[\begin{array}{c} \mathfrak{b} \\ \omega \end{array} \right]_{\aleph_0}^{1,1}$$

This answers [016GS, Problems 3.6 & 3.10] negatively.

Proof. Assume that $\mathfrak{b} = \mathfrak{d}$. Then we may construct an unbounded scale $\langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$ in ${}^\omega\omega$ and assume w.l.o.g. that for every $\alpha < \mathfrak{b}$ the sequence f_α is properly increasing and $f_\alpha(0) \in \omega \setminus 1$. Furthermore associate to each countable ordinal α a function $g_\alpha : \omega \rightarrow \omega$ defined inductively by $g_\alpha(0) := 0$ and $g_\alpha(n+1) := f_\alpha(g_\alpha(n))$. Note that for each countable ordinal α , the function g_α is properly increasing. Fix a mapping $s : \omega \rightarrow \omega$ such that for all natural numbers n the set $s^{-1}[n]$ is infinite. Now we define a colouring χ as follows:

$$(4) \quad \begin{aligned} \chi : \mathfrak{b} \times \omega &\longrightarrow \omega, \\ \langle \alpha, k \rangle &\longmapsto s\left(\min(\{n < \omega \mid k \leq g_\alpha(n)\})\right) \end{aligned}$$

Suppose towards a contradiction that $X \in [\mathfrak{b}]^{\mathfrak{b}}$ and $Y \in [\omega]^\omega$ are such that $\chi[X \times Y] \neq \omega$. Consider the function

$$(5) \quad \begin{aligned} h : \omega &\longrightarrow \omega, \\ n &\longmapsto \min(Y \setminus (n+1)). \end{aligned}$$

Note that for all natural numbers n we have $h(n) > n$. Let $i \in \omega \setminus \chi[X \times Y]$. Choose an $\alpha \in X$ such that f_α properly dominates h (here we need that $\langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$ is a scale). Let j be a sufficiently large natural number such that for all $n \in \omega \setminus j$ we have $f_\alpha(n) > h(n)$. Choose a $k \in s^{-1}[i] \setminus (j+1)$ and define $\ell := \max(Y \cap g_\alpha(k))$. But then we have

$$(6) \quad s\left(\min(\{n < \omega \mid \ell \leq g_\alpha(n)\})\right) = \chi(\alpha, \ell) \neq i.$$

Now this implies that there is an $m < k$ with $\ell \leq g_\alpha(m)$ which in turn gives rise to the following:

$$(7) \quad Y \setminus (\ell+1) \ni h(\ell) \leq h(g_\alpha(m)) \leq h(g_\alpha(k-1)) < f_\alpha(g_\alpha(k-1)) = g_\alpha(k).$$

This contradicts the definition of ℓ thus proving the Theorem. □

Theorem 2.2 solves [016GS, Problem 3.19]. The problem asks whether for a cardinal $\aleph_1 < \kappa < \mathfrak{c}$ it is possible for $\binom{\kappa}{\omega} \longrightarrow \binom{\kappa}{\omega}_2^{1,1}$ to be destroyed by the Lévy collapse of \mathfrak{c} to κ . We give an affirmative answer by showing that it is consistent that $\aleph_1 < \kappa < \mathfrak{c}$ and $\binom{\kappa}{\omega} \longrightarrow \binom{\kappa}{\omega}_2^{1,1}$ but after Lévy-collapsing \mathfrak{c} to κ one has $\binom{\kappa}{\omega} \not\rightarrow \binom{\kappa}{\omega}_2^{1,1}$.

Fischer and Steprāns proved in [008FS], that $\aleph_1 < \mathfrak{b} < \mathfrak{s}$ is consistent. Lévy-collapsing \mathfrak{c} to \mathfrak{b} over this model yields a model with a scale of length \mathfrak{b} which by Theorem 2.2 implies $\binom{\mathfrak{b}}{\omega} \not\rightarrow \binom{\mathfrak{b}}{\omega}$.

Note that this has been solved independently by Garti and Shelah in [018GS].

An obvious question is whether or not the hypothesis in Theorem 2.2 is necessary. The answer to this question is negative.

Theorem 2.3. *It is consistent that $\mathfrak{b} < \mathfrak{d}$ and $\binom{\mathfrak{d}}{\omega} \rightarrow [\binom{\omega_1}{\omega}]_{\aleph_0}^{1,1}$. This amounts also to $\binom{\mathfrak{d}}{\omega} \rightarrow [\binom{\mathfrak{b}}{\omega}]_{\aleph_0}^{1,1}$. The distance between \mathfrak{b} and \mathfrak{d} can be arbitrarily large.*

Proof. We assume the continuum hypothesis in the ground model, and we choose $\lambda > \aleph_1$ so that $\lambda^\omega = \lambda$. Let \mathbb{Q} be the following forcing notion. We say that $p \in \mathbb{Q}$ iff p is a partial function from ω into ω and $|\text{dom}(p)| < \aleph_0$. If $p, q \in \mathbb{Q}$ then $p \leq q$ iff $p \subseteq q$.

Let \mathbb{P} be the finite support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$, when \mathbb{Q}_β is a \mathbb{P}_β -name of the forcing \mathbb{Q} at every successor stage. It is known that \mathbb{Q} is isomorphic to the usual Cohen forcing (see, e.g., Halbeisen, Combinatorial set theory) and hence \mathbb{P} is isomorphic to the forcing which adds λ many Cohen reals. In particular, \mathbb{P} is *ccc* so no cardinal is collapsed and no cofinality is changed.

Let $G \subseteq \mathbb{P}$ be generic. It follows that $V[G] \models \mathfrak{b} = \aleph_1 < \lambda = \mathfrak{d} = \mathfrak{c}$. We shall prove that $V[G] \models \binom{\lambda}{\omega} \rightarrow [\binom{\omega_1}{\omega}]_{\aleph_0}^{1,1}$. For this end, let $(f_\alpha : \alpha < \lambda)$ be an enumeration of the Cohen reals we added, so $f_\alpha \in {}^\omega \omega$ for every $\alpha < \lambda$. We define a name \underline{c} of a coloring from $\lambda \times \omega$ into \aleph_0 as follows. Given $\alpha < \lambda$ and $n \in \omega$ we let $\underline{c}(\alpha, n) = f_\alpha(n)$. We claim that \underline{c} exemplifies the negative relation to be proved.

For this, assume that $\underline{A} \in [\lambda]^{\aleph_1}$ and $\underline{B} \in [\omega]^\omega$. Assume towards contradiction that there exists a condition $p_0 \in \mathbb{P}$ which forces that $\underline{c}(\underline{A} \times \underline{B}) \neq \aleph_0$. We may extend p_0 into a condition p such that $p \Vdash \check{m} \notin \underline{c}(\underline{A} \times \underline{B})$ for some $m \in \omega$. The idea of the proof is to find some $\alpha \in \lambda$ and $n \in \omega$ for which no value has been fixed yet, to force them into $\underline{A} \times \underline{B}$ and then to assign the value m to $\underline{c}(\alpha, n)$.

Firstly, for every $n \in \omega$ let \mathcal{A}_n be a maximal anti-chain which decides the statement $\check{n} \in \mathcal{B}$. The size of each \mathcal{A}_n is at most \aleph_0 , since \mathbb{P} is *ccc*. Likewise, $|\text{dom}(q)| < \aleph_0$ for every $q \in \mathcal{A}_n$, so the set $U = \bigcup\{\text{dom}(q) : q \in \mathcal{A}_n, n \in \omega\}$ is countable. Recall that $\check{A} \in [\lambda]^{\aleph_1}$, so $\Vdash_{\mathbb{P}} \check{A} \not\subseteq U$. Fix any ordinal $\alpha \in \lambda$ so that $p \Vdash \check{\alpha} \notin \check{A}$. Now we choose a condition $q \geq p$ such that $q \Vdash \check{\alpha} \in \check{A}$, and we may assume without loss of generality that $\alpha \in \text{dom}(q)$.

Secondly, we need the appropriate $n \in \omega$. Choose $n_0 \in \omega$ such that $\text{sup}(\text{dom}(q(\alpha))) < n_0$. We choose some $n \in \omega$ and a condition $r \geq q \upharpoonright U$ such that $\text{dom}(r) \subseteq U$ and $r \Vdash n_0 < n \in \mathcal{B}$. Let s be $r \cup q \upharpoonright (\text{dom}(q) \setminus U)$ and let t be $s \cup \langle \alpha, n, m \rangle$. It follows that $t \Vdash \mathcal{C}(\alpha, n) = m$, a contradiction. \square

Now we use Theorem 2.2 to settle another question from [018GS]. Recall from Fact 1.2 that if $\mathfrak{r} < \kappa \leq \mathfrak{c}$ and $\text{cf}(\kappa) > \mathfrak{r}$, then $\binom{\kappa}{\omega} \longrightarrow \binom{\kappa}{\omega}$. Asking what happens when $\aleph_0 < \text{cf}(\kappa) \leq \mathfrak{r}$ gives rise to the following problem, cf. [018GS, Question 1.8(a)]:

Question 2.4. Is it consistent that $2^{\aleph_0} = \lambda > \mathfrak{r}$, $\text{cf}(\lambda) > \aleph_0$ and $\binom{\lambda}{\omega} \not\rightarrow \binom{\lambda}{\omega}_2^{1,1}$?

We shall give a negative answer, which is a bit surprising. It demonstrates the fact that the duality between \mathfrak{r} and \mathfrak{s} is not totally complete. Recall that \mathfrak{i} , the independence number, is the minimal size of a maximal independent family in $[\omega]^\omega$. It is well known that $\mathfrak{r}, \mathfrak{d} \leq \mathfrak{i}$, see e.g. [010B1]. It is also known that one can force $\mathfrak{i} = \aleph_1$ and $\lambda = 2^{\aleph_0}$ provided that $\text{cf}(\lambda) > \omega$.

Theorem 2.5. *One can force $\mathfrak{r} < \lambda$, $\text{cf}(\lambda) > \omega$ and yet $\binom{\lambda}{\omega} \not\rightarrow \binom{\lambda}{\omega}_2^{1,1}$.*

Proof. Let us prove the following general assertion.

Claim 2.6. If $\binom{\kappa}{\omega} \not\rightarrow \binom{\kappa}{\omega}_2^{1,1}$ and $\lambda > \text{cf}(\lambda) = \kappa$ then $\binom{\lambda}{\omega} \not\rightarrow \binom{\lambda}{\omega}_2^{1,1}$.

Proof of Claim. For this, choose a continuous increasing sequence of cardinals $\langle \vartheta_\varepsilon : \varepsilon < \kappa \rangle$ such that $\lambda = \bigcup_{\varepsilon < \kappa} \vartheta_\varepsilon$ and $\vartheta_{\varepsilon+1}$ is a regular cardinal for every $\varepsilon < \kappa$. Define the intervals mapping $h : \lambda \rightarrow \kappa$ by $h(\alpha) = \min\{\varepsilon < \kappa : \vartheta_\varepsilon \leq \alpha < \vartheta_{\varepsilon+1}\}$.

Choose a colouring $c : \kappa \times \omega \rightarrow 2$ which exemplifies the negative relation $\binom{\kappa}{\omega} \not\rightarrow \binom{\kappa}{\omega}_2^{1,1}$. For every $\alpha < \lambda$, $n \in \omega$ let $d(\alpha, n)$ be $c(h(\alpha), n)$, so $d : \lambda \times \omega \rightarrow 2$. We claim that d exemplifies the negative relation $\binom{\lambda}{\omega} \not\rightarrow \binom{\lambda}{\omega}_2^{1,1}$.

Assume toward contradiction that $A \in [\lambda]^\lambda$, $B \in [\omega]^\omega$ and $d \upharpoonright (A \times B)$ is constant. Let $H = \{h(\alpha) : \alpha \in A\}$. Since A is unbounded in λ and $\text{cf}(\lambda) = \kappa$ we see that $H \in [\kappa]^\kappa$. We shall prove that $c \upharpoonright (H \times B)$ is constant, thus arriving at a contradiction.

Let j be the constant value of d over $A \times B$. Choose any $\varepsilon \in H, n \in \omega$. Let α be an element of A such that $\varepsilon = h(\alpha)$. From the definition of c it follows that $c(\varepsilon, n) = c(h(\alpha), n) = d(\alpha, n) = j$, so the claim is proved. \textcircled{C}

Now force $\mathfrak{i} = \aleph_1$ while $\lambda = 2^{\aleph_0} = \aleph_{\omega_1}$. As noted above, $\mathfrak{d} \leq \mathfrak{i}$ hence $\mathfrak{b} = \mathfrak{d} = \omega_1$ so $(\omega_1) \not\rightarrow (\omega_1)_2^{1,1}$. By the above statement, $(\lambda) \not\rightarrow (\lambda)_2^{1,1}$ as well, so the proof is accomplished. \square

Theorem 2.2 can be improved if $\mathfrak{d} = \aleph_1$.

Theorem 2.7. *Suppose that $\mathfrak{d} = \aleph_1$. Then*

$$(\omega_1) \not\rightarrow \left[\begin{array}{c} \omega_1 \\ \omega \vee \omega_1 \end{array} \right]_{\aleph_0}^{1,1}.$$

Proof. We use the following lemma without proof which comes from the construction of an Aronszajn tree.

Lemma 2.8. *There exists a sequence of functions $\langle g_\alpha : \alpha < \omega_1 \rangle$ so that for all countable ordinals α :*

- (1) $g_\alpha : \alpha \rightarrow \omega$ is injective.
- (2) For any countable ordinal δ , the set $\{g_\alpha \upharpoonright \delta \mid \alpha < \omega_1\}$ is countable.

Fix a sequence of functions $\langle g_\alpha : \alpha < \omega_1 \rangle$ as in Lemma 2.8.

Let $s : \omega \rightarrow \omega$ be such that for all natural numbers i the set $s^{-1}[\{i\}]$ is infinite. Moreover, let $\langle f_\alpha \mid \alpha < \omega_1 \rangle$ be a scale. We may assume w.l.o.g. that $f_\alpha(n) > n$ for all countable ordinals α and all natural numbers n . Define a sequence $\langle k_\alpha \mid \alpha < \omega_1 \rangle$ by setting $(k_{\omega\alpha+i+1} := f_\alpha(k_{\omega\alpha+i})$ for all natural numbers i and $k_{\omega\alpha} := 0)$ for all countable ordinals α . Note that for every countable ordinal α the sequence $\langle k_{\omega\alpha+i} \mid i < \omega \rangle$ is strictly increasing. Now we define a colouring as follows:

$$(8) \quad \chi : \omega_1 \times \omega_1 \longrightarrow \omega,$$

$$\langle \alpha, \beta \rangle \longmapsto \begin{cases} s(\min \{i < \omega \mid g_\alpha(\beta) < k_{\omega\alpha+i}\}) & \text{if } \beta < \alpha, \\ \chi(\beta, \alpha) & \text{if } \alpha < \beta, \\ 0 & \text{else.} \end{cases}$$

Now arbitrarily choose sets $X \in [\omega_1]^{\omega_1}$ and $Y \in [\omega_1]^\omega$ as well as a colour n . Let $\gamma := \sup Y$. By condition (2) of the statement of Lemma 2.8, the set $H := \{g_\delta \upharpoonright \gamma \mid \delta < \omega_1\}$ is countable. Let $\langle h_i \mid i < \omega \rangle$ be an enumeration of H . We define a function

$$(9) \quad \begin{aligned} h : \omega &\longrightarrow \omega \\ k &\longmapsto \max_{i < k} (\min \{j \in \omega \setminus k \mid h_i^{-1}(j) \in Y\}) \end{aligned}$$

Let $\alpha \in X \setminus (\gamma + 1)$ be sufficiently large such that f_α eventually properly dominates h . Let ℓ be such that $h_\ell = g_\alpha \upharpoonright \gamma$. Let $\bar{\ell} \in \omega \setminus (\ell + 2)$ be such that $f_\alpha(i) > h(i)$ for all $i \in \omega \setminus \bar{\ell}$ and choose an $m \in \omega \setminus \bar{\ell}$ such that $s(m) = n$. We have

$$(10) \quad k_{\omega\alpha+m} = f_\alpha(k_{\omega\alpha+m-1}) > h(k_{\omega\alpha+m-1}) \geq \min \{j \in \omega \setminus k_{\omega\alpha+m-1} \mid h_\ell^{-1}(j) \in Y\}.$$

In other words, there is an ordinal $\beta \in Y$ such that $h_\ell(\beta) = g_\alpha(\beta) \in k_{\omega\alpha+m} \setminus k_{\omega\alpha+m-1}$. But then, clearly, $\chi(\alpha, \beta) = s(m) = n$. \square

With a creature forcing construction, Shelah proved that it is consistent that $\aleph_1 = \mathfrak{b} < \mathfrak{s}$. Therefore, the hypothesis of Theorem 2.7 cannot be weakened to $\mathfrak{b} = \aleph_1$, i.e., it is consistent that $\mathfrak{b} = \aleph_1$ and $\binom{\omega_1}{\omega} \rightarrow \binom{\omega_1}{\omega}_2^{1,1}$.

We answer Jean Larson's question from [998La, page 112] affirmatively.

Theorem 2.9. *Suppose that κ is regular and $\lambda = \kappa^+ = \mathfrak{d}_\kappa$. Then $\lambda^2 \not\rightarrow (\lambda\kappa, 4)^2$.*

Proof. For every ordinal $\xi < \lambda$ we fix the following things:

$$(11) \quad \text{An injection } b_\xi : \xi \hookrightarrow \kappa,$$

$$(12) \quad \text{an increasing function } d_\xi : \kappa \longrightarrow \kappa, \text{ and}$$

$$(13) \quad \text{sequences of ordinals } e(\xi, \cdot), f(\xi, \cdot) : \kappa \longrightarrow \xi.$$

More specifically we do this in a way such that

$$(14) \quad \{d_\xi \mid \xi < \lambda\} \text{ is a scale of length } \lambda \text{ in } {}^\kappa\kappa \text{ such that}$$

$$(15) \quad \forall \xi < \lambda \forall \nu < \kappa (d_\xi(\nu) > \nu) \text{ and}$$

$$(16) \quad g_\xi : \kappa \longrightarrow \xi \times \xi,$$

$$(17) \quad \nu \longmapsto \langle e(\xi, \nu), f(\xi, \nu) \rangle$$

$$(18) \quad \text{satisfies } \forall x \in \xi \times \xi : |g_\xi^{-1}[x]| = \kappa.$$

We are going to define a graph $\langle \lambda^2, \Delta \rangle$ where $\{\lambda\alpha + \beta, \lambda\gamma + \delta\} \in \Delta$ together with $\lambda\alpha + \beta < \lambda\gamma + \delta$ implies $\alpha < \gamma < \delta < \beta$. (Note that every graph corresponds to a 2-colouring of its vertex-set where a pair of vertices gets one colour if both vertices are connected to each other by an edge and the other if they are not.) Given a $\beta < \lambda$ and an $\alpha < \beta$ we inductively define the sets $\Gamma(\lambda\alpha + \beta) = \{\lambda\gamma + \delta \mid \alpha \leq \gamma \leq \delta \leq \beta \wedge \{\lambda\alpha + \beta, \lambda\gamma + \delta\} \in \Delta\}$. The induction is layered. The top layer of the induction has length λ and in step $\beta < \lambda$ we define all sets $\Gamma(\lambda\alpha + \beta)$ with $\alpha < \beta$. The second layer of the induction has length β and in step $\alpha < \beta$ we define the set $\Gamma(\lambda\alpha + \beta)$. The third layer of the induction has length κ where in every step fewer than κ ordinals are added to $\Gamma(\lambda\alpha + \beta)$ as elements.

Suppose that we are in step ζ of the third layer of the induction and previous steps $\mu < \zeta$ have added the ordinals $\lambda\gamma_{\mu,\nu} + \delta_{\mu,\nu,\xi}$ ($\nu < \vartheta_\mu, \xi < \iota_{\mu,\nu}$) to $\Gamma(\lambda\alpha + \beta)$ and have defined σ_μ . As induction hypothesis we assume that for all $\mu < \zeta$ we have ($\vartheta_\mu < \kappa$ and $\forall \nu < \vartheta_\mu : \iota_{\mu,\nu} < \kappa$). Using κ 's regularity we now choose a $\rho_\zeta < \kappa$ satisfying

$$(19) \quad \rho_\zeta > \sup_{\xi < \zeta} \sigma_\xi,$$

$$(20) \quad b_{e(\beta,\zeta)}^{-1}(\rho_\zeta) > \sup(e(\beta, \zeta) \cap \{\gamma_{\mu,\nu} \mid \mu < \zeta \wedge \nu < \vartheta_\mu\}),$$

$$(21) \quad b_{e(\beta,\zeta)}^{-1}(\rho_\zeta) > \sup(\{b_{\gamma_{\mu,\nu}}^{-1}[b_{\gamma_{\mu,\nu}}(\alpha)] \mid \mu < \zeta \wedge \nu < \vartheta_\mu \wedge e(\beta, \zeta) \leq \gamma_{\mu,\nu}\}),$$

$$(22) \quad b_{e(\beta,\zeta)}^{-1}(\rho_\zeta) > \sup(\{b_{\gamma_{\mu,\nu}}^{-1}[b_{\gamma_{\mu,\nu}}(\gamma_{\mu',\nu'})] \mid \mu, \mu' < \zeta \wedge \nu, \nu' < \vartheta_\mu \wedge e(\beta, \zeta) \leq \gamma_{\mu',\nu'} < \gamma_{\mu,\nu}\}).$$

and set $\sigma_\zeta := d_\beta(\rho_\zeta)$. Note that together with (15) and (19) this implies that

$$(23) \quad \forall \xi < \kappa (\xi < \rho_\xi < \sigma_\xi).$$

Now let $\langle \gamma_{\zeta, \nu} \mid \nu < \vartheta_\zeta \rangle$ be the increasing enumeration of the set

$$(24) \quad \{b_{e(\beta, \zeta)}^{-1}(\xi) \mid \xi \in \sigma_\zeta \setminus \rho_\zeta \wedge b_{e(\beta, \zeta)}^{-1}(\rho_\zeta) \leq b_{e(\beta, \zeta)}^{-1}(\xi)\}$$

Now inductively for every $\mu < \vartheta_\zeta$ let

$$(25) \quad \varphi_{\zeta, \mu} := \sup \{ \max(\sigma_\zeta, b_{f(\beta, \zeta)}(\delta)) \mid \exists \nu, \xi ((\nu < \zeta \wedge \xi < \vartheta_\nu) \vee (\nu = \zeta \wedge \xi < \mu)) \wedge \exists \tau < \iota_{\nu, \xi} : \lambda \gamma_{\zeta, \mu} + \delta \in \Gamma(\lambda \gamma_{\nu, \xi} + \delta_{\nu, \xi, \tau}) \},$$

let $\iota_{\zeta, \mu}$ be the least ordinal ι such that $\varphi_{\zeta, \mu} + \iota \geq d_\beta(\varphi_{\zeta, \mu})$ and $\delta_{\zeta, \mu, \nu} := b_{f(\beta, \zeta)}^{-1}(\varphi_{\zeta, \mu} + \nu)$ for every $\nu < \iota_{\zeta, \mu}$.

This finishes the definition of $\Gamma(\lambda\alpha + \beta)$ and thereby the definition of Δ . Now we check that Δ witnesses $\lambda^2 \not\rightarrow (\lambda\kappa, 4)^2$.

First assume that there was a $Q \in [\lambda^2]^4$ such that $[Q]^2 \subseteq \Delta$. Suppose that $Q = \{\lambda\alpha_0 + \beta_0, \lambda\alpha_1 + \beta_1, \lambda\alpha_2 + \beta_2, \lambda\alpha_3 + \beta_3\}$ with $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$. Then, w.l.o.g., we get $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \beta_3 < \beta_2 < \beta_1 < \beta_0$. Now first assume that there are $\ell \in 4 \setminus 2$ and $k \in \ell \setminus 1$ such that $\{\lambda\alpha_0 + \beta_0, \lambda\alpha_k + \beta_k\}$ was added to Δ before $\{\lambda\alpha_0 + \beta_0, \lambda\alpha_\ell + \beta_\ell\}$ —that is, $\lambda\alpha_k + \beta_k$ was added to $\Gamma(\lambda\alpha_0 + \beta_0)$ before $\lambda\alpha_\ell + \beta_\ell$. So assume that $\lambda\alpha_k + \beta_k$ was added in induction step ζ and $\lambda\alpha_\ell + \beta_\ell$ was added in induction step μ where $\zeta \leq \mu < \kappa$. Let $\nu < \vartheta_\zeta$ be such that $\gamma_{\zeta, \nu} = \alpha_k$ and let $\xi < \iota_{\zeta, \nu}$ be such that $\delta_{\zeta, \nu, \xi} = \beta_k$. Similarly, let $\tau < \vartheta_\mu$ be such that $\gamma_{\mu, \tau} = \alpha_\ell$ and let $\psi < \iota_{\mu, \tau}$ be such that $\delta_{\mu, \tau, \psi} = \beta_\ell$. Then $b_{f(\beta, \mu)}(\beta_\ell) = \varphi_{\mu, \tau} + \psi$. As

$$(26) \quad \lambda \gamma_{\mu, \tau} + \delta_{\mu, \tau, \psi} = \lambda \alpha_\ell + \beta_\ell \in \Gamma(\lambda \alpha_k + \beta_k) = \Gamma(\lambda \gamma_{\zeta, \nu} + \delta_{\zeta, \nu, \xi})$$

$$(27) \quad \text{and either } (\zeta < \mu \wedge \nu < \vartheta_\zeta) \text{ or } (\zeta = \mu \wedge \nu < \tau), \text{ by (25) we have}$$

$$(28) \quad b_{f(\beta, \mu)}(\beta_\ell) = b_{f(\beta, \mu)}(\delta_{\mu, \tau, \psi}) = \varphi_{\mu, \tau} + \psi \geq \varphi_{\mu, \tau} > b_{f(\beta, \mu)}(\delta_{\mu, \tau, \psi}) = b_{f(\beta, \mu)}(\beta_\ell),$$

a contradiction.

Now assume that this is not the case, i.e. for $k \in 4 \setminus 1$ the set $\lambda\alpha_k + \beta_k$ was added to $\Gamma(\lambda\alpha_0 + \beta_0)$ in induction step μ_k where $\mu_3 \leq \mu_2 \leq \mu_1 < \kappa$. Also, for $k \in 4 \setminus 1$, let $\nu_k < \vartheta_{\mu_k}$ be such that $\alpha_k = \gamma_{\mu_k, \nu_k}$. Then, for $k \in \{1, 2\}$, if $\mu_k = \mu_{k+1}$ then

$\nu_{k+1} < \nu_k$. We have

$$(29) \quad \mu_3 < \mu_2 < \mu_1 < \kappa$$

because if for some $k \in \{1, 2\}$ we have $\mu_k = \mu_{k+1}$, then $\alpha_{k+1} = \gamma_{\mu_{k+1}, \nu_{k+1}} = \gamma_{\mu_k, \nu_{k+1}} < \gamma_{\mu_k, \nu_k} = \alpha_k$, a contradiction. This proves (29).

Furthermore we have

$$(30) \quad e(\beta_\ell, \mu_k) \leq \alpha_{k+1} \text{ for both } k \in \{1, 2\} \text{ and } \ell < k$$

since if there was a $k \in \{1, 2\}$ with $e(\beta_\ell, \mu_k) > \alpha_{k+1} = \gamma_{\mu_{k+1}, \nu_{k+1}}$, then by (20), $b_{e(\beta_\ell, \mu_k)}^{-1}(\rho_{\mu_k}) > \gamma_{\mu_{k+1}, \nu_{k+1}}$. By (24), we have $\gamma_{\mu_k, \nu_k} \geq b_{e(\beta_\ell, \mu_k)}^{-1}(\rho_{\mu_k})$ and hence $\alpha_k = \gamma_{\mu_k, \nu_k} > \gamma_{\mu_{k+1}, \nu_{k+1}} = \alpha_{k+1}$, a contradiction. This proves (30).

Now we get

$$(31) \quad b_{\alpha_3}(\alpha_2) < b_{\alpha_3}(\alpha_1)$$

as otherwise $b_{\alpha_3}(\gamma_{\mu_2, \nu_2}) = b_{\alpha_3}(\alpha_2) \geq b_{\alpha_3}(\alpha_1) = b_{\alpha_3}(\gamma_{\mu_1, \nu_1})$ and as b_{α_3} is a bijection, $b_{\alpha_3}(\gamma_{\mu_2, \nu_2}) > b_{\alpha_3}(\gamma_{\mu_1, \nu_1})$. Then, by (22) in combination with (30) for $\ell = 0$, we have $b_{e(\beta_0, \mu_1)}^{-1}(\rho_{\mu_1}) > \gamma_{\mu_1, \nu_1}$. But by (24), we have $\gamma_{\mu_1, \nu_1} \geq b_{e(\beta_0, \mu_1)}^{-1}(\rho_{\mu_1})$, a contradiction. This proves (31).

Now, by (21) in combination with (30) for $\ell = 1$ and $k = 2$ and (31), we have $b_{e(\beta_1, \mu_2)}^{-1}(\rho_{\mu_2}) > \alpha_2$. But then, by (24), we get $\alpha_2 \geq b_{e(\beta_1, \mu_2)}^{-1}(\rho_{\mu_2})$, a contradiction.

This refutes the existence of a $Q \in [\lambda^2]^4$ such that $[Q]^2 \subseteq \Delta$.

Now assume that there is an $H \in [\lambda^2]^{\lambda^\kappa}$ such that $[H]^2 \subseteq [\lambda^2]^2 \setminus \Delta$. We define

$$(32) \quad A := \{\mu < \lambda \mid |\{\nu < \lambda \mid \lambda\mu + \nu \in H\}| = \lambda\}.$$

It is easy to show that $A \in [\lambda]^\kappa$. Let $\xi := \sup(A)$ and $\tau := \min \{\mu \in \lambda \setminus \xi \mid \forall \nu \in A : |\{\psi < \mu \mid \lambda\nu + \psi \in H\}| = \kappa\}$. Now we define two functions

$$(33) \quad h_0 : \kappa \longrightarrow \kappa$$

$$(34) \quad \mu \longmapsto \min \{\nu \in \kappa \setminus \mu \mid b_\xi^{-1}(\nu) \in A\}$$

$$(35) \quad h_1 : \kappa \longrightarrow \kappa$$

$$(36) \quad \mu \longmapsto \min \{\nu \in \kappa \setminus \mu \mid \forall \gamma \in A (b_\xi(\gamma) \leq \mu \rightarrow \exists \delta (b_\tau(\delta) \in \nu \setminus \mu \wedge \lambda\gamma + \delta \in H))\}.$$

Let $\alpha := \min(A)$. Moreover, let $\beta \in \lambda \setminus \max(\xi, \tau)$ be such that d_β eventually dominates h_i for both $i < 2$ and $\lambda \cdot \alpha + \beta \in H$. Let $\mu < \kappa$ be such that for both $i < 2$ the sequence d_β properly dominates h_i above μ . Let $\zeta \in \kappa \setminus (\max(\mu, b_\xi(\alpha)) + 1)$ such that $g_\beta(\zeta) = \langle \xi, \tau \rangle$. We have $\rho_\zeta \geq \zeta$. As d_β properly dominates h_0 at ρ_ζ , the definition of h_0 implies $\sigma_\zeta = d_\beta(\rho_\zeta) > \min \{\nu \in \kappa \setminus \rho_\zeta \mid b_\xi^{-1}(\nu) \in A\}$. So there is a $\nu < \vartheta_\zeta$ such that $\gamma_{\zeta, \nu} \in A$, cf. (24).

Also, $\varphi_{\zeta, \nu} \geq \sigma_\zeta \geq \rho_\zeta \geq \zeta > \mu$. Therefore, $\varphi_{\zeta, \nu} + \iota_{\zeta, \nu} = d_\beta(\varphi_{\zeta, \nu}) > h_1(\varphi_{\zeta, \nu})$. As $b_\xi(\gamma_{\zeta, \nu}) < \sigma_\zeta \leq \varphi_{\zeta, \nu}$ there is a $\delta < \tau$ such that $\mu \leq b_\tau(\delta) < h_1(\varphi_{\zeta, \nu}) < d_\beta(\varphi_{\zeta, \nu})$ and $\lambda\gamma_{\zeta, \nu} + \delta \in H$. But this means that there is a $\psi < \iota_{\zeta, \nu}$ such that $\delta = \delta_{\zeta, \nu, \psi}$ and $\lambda\gamma_{\zeta, \nu} + \delta_{\zeta, \nu, \psi} \in H \cap \Gamma(\lambda\alpha + \beta) \subseteq \Delta$. As $\lambda\alpha + \beta \in H$ we have $\{\lambda\alpha + \beta, \lambda\gamma_{\zeta, \nu} + \delta_{\zeta, \nu, \psi}\} \in [H]^2$ showing $[H]^2 \not\subseteq [\lambda^2]^2 \setminus \Delta$, a contradiction. \square

Corollary 2.10. *If κ is regular and $\lambda = \kappa^+ = \mathfrak{d}_\kappa$, then $\alpha \not\rightarrow (\lambda\kappa, 4)^2$ for all $\alpha < \lambda^2\kappa$.*

3. PARTITION RELATIONS FROM THE STICK-PRINCIPLE

The principle $\blacklozenge(\kappa) = \kappa^+$ can be used directly to guess an infinite subset of a homogeneous set. In some cases, a diagonal process which takes care of all of the guesses can be used to obtain a negative partition relation. This is the idea behind Takahashi's result that $\blacklozenge = \aleph_1$ implies $\omega_1^2 \not\rightarrow (\omega_1^2, 3)^2$ (and the result generalizes straightforwardly to arbitrary κ).

Combining this method with the argument of Todorćević [989To] gives the following:

Theorem 3.1. *Let κ be a regular cardinal. Then $\blacklozenge(\kappa) = \kappa^+$ implies $\kappa^+ \not\rightarrow (\kappa^+, (\kappa : 2))^2$.*

Proof. In the proof below, we use the adverb *almost* to mean “modulo sets of size $< \kappa$ ”.

Let $\langle e_\alpha : \alpha < \kappa^+ \rangle$ be a sequence of order-type κ subsets witnessing $\mathfrak{p}(\kappa) = \kappa^+$. Let $\langle d_\alpha \mid \alpha < \kappa^+ \rangle$ be an enumeration of $\left\{ \bigcup_{\beta \in e_\gamma} e_\beta \mid \gamma < \kappa^+ \right\}$ so that $d_\alpha \subseteq \alpha$. Note that this also witnesses $\mathfrak{p} = \kappa^+$. Define by induction on $\alpha < \kappa^+$ a sequence $\langle B_\alpha \mid \alpha < \kappa^+ \rangle$ of size κ subsets of κ^+ so that:

- (1) $B_\alpha \subseteq \alpha$.
- (2) $|B_\alpha \cap B_{\alpha'}| < \kappa$ if $\alpha \neq \alpha'$.
- (3) $B_\alpha \cap d_\beta \neq \emptyset$ if $\beta < \alpha$ and d_β is not contained in any $< \kappa$ -union of sets of the form $B_{\alpha'}$ for $\alpha' < \alpha$.

For the construction at stage α , let $g_\alpha : \kappa \rightarrow \alpha$ be a surjection. Then inductively construct B_α to be the set $\{x_i^\alpha \mid i < \kappa\}$, where

$$x_i^\alpha \in d_{g_\alpha(i)} \setminus \bigcup_{j < i} B_{g_\alpha(j)}$$

if such exists.

Let the coloring be defined by $c(\alpha', \alpha) = 1$ iff $\alpha' < \alpha$ and $\alpha' \in B_\alpha$. There is no 1-homogenous set $\{\alpha_\xi \mid \xi < \kappa + 2\}$ of type $(\kappa : 2)$ since otherwise $\{\alpha_\xi \mid \xi < \kappa\} \subseteq B_{\alpha_\kappa} \cap B_{\alpha_{\kappa+1}}$, contradicting (2) in the construction of the B_α 's.

Suppose there is a 0-homogeneous set A of size κ^+ . In the construction we ensured that the B_α 's met d_α often enough so that the following claim gives a contradiction.

Claim 3.2. There is $\beta < \kappa^+$ so that $d_\beta \subseteq A$ and d_β is not almost contained in any $< \kappa$ -union of sets of the form $B_{\alpha'}$ for $\alpha' < \kappa^+$.

Proof of Claim. The proof of the claim splits into two cases depending on the cardinality of $C := \{\alpha \mid |B_\alpha \cap A| = \kappa\}$.

The first case is that $|C| \leq \kappa$. Then $|A \setminus \bigcup_{\alpha \in C} B_\alpha| > \kappa$, so there is some $\beta < \kappa^+$ with $d_\beta \subseteq A \setminus \bigcup_{\alpha \in C} B_\alpha$. This case is finished by observing that d_β has intersection of size $< \kappa$ with every B_α .

The second case is that $|C| = \kappa^+$. For each $i < \kappa^+$, define $\xi(i)$ and $\zeta(i)$ so that

- $e_{\xi(i)} \subseteq \bigcup_{\alpha \in C} B_\alpha \cap A$.
- $|B_{\zeta(i)} \cap e_{\xi(i)}| = \kappa$, if such exists, and $\zeta(i)$ is undefined otherwise.
- $e_{\xi(i)} \cap \bigcup_{j < i} (B_{\zeta(j)} \cup e_{\xi(j)}) = \emptyset$.

There is some $\gamma < \kappa^+$ so that $e_\gamma \subseteq \{\xi(i) \mid i < \kappa^+\}$. Let β be so that $d_\beta = \bigcup_{\xi \in e_\gamma} e_\xi$ and let $I = \{i < \kappa^+ \mid \xi(i) \in e_\gamma\}$.

Suppose for a contradiction that d_β is almost contained in $\bigcup_{\nu < \eta} B_{\alpha_\nu}$ for some α_ν , $\nu < \eta$. There are two subcases.

In the first subcase, there are κ many $i \in I$ on which $\zeta(i)$ is defined. For each such i , $|B_{\zeta(i)} \cap e_{\xi(i)}| = \kappa$ so there is some $k(i) < \eta$ so that $|B_{\zeta(i)} \cap B_{\alpha_{k(i)}}| = \kappa$, and since the B_α are almost disjoint, $\zeta(i) = \alpha_{k(i)}$. But this is a contradiction since $|I| = \kappa$ and ζ is injective, but the range of $i \mapsto k(i)$ has size $< \kappa$.

In the second subcase, there are $< \kappa$ many $i \in I$ on which $\zeta(i)$ is defined, so we can take $\xi^* \in e_\gamma$ so that $|e_{\xi^*} \cap B_{\alpha_k}| < \kappa$ for all $k < n$. But this implies that $|e_{\xi^*}| < \kappa$, contradiction. \textcircled{c}

To finish the proof of the theorem, take $\alpha \in A$ larger than β . By (3) in the construction of the B_α 's, there is some α' with $\alpha' \in B_\alpha \cap d_\beta$. Now $\alpha' \in d_\beta \subseteq \beta < \alpha$, so $c(\alpha', \alpha) = 1$. Since $d_\beta \subseteq A$, we have $\alpha', \alpha \in A$, contradicting 0-homogeneity of A . \square

We now give a few remarks regarding the relationship between the $\overset{\bullet}{\uparrow}$ principle and polarised partition relations. We conjecture that the hypothesis of Theorem 2.7 cannot be changed to $\overset{\bullet}{\uparrow} = \aleph_1$. Curiously, however, $\overset{\bullet}{\uparrow} = \aleph_1$ actually gives a polarised partition relation involving larger cardinals.

Proposition 3.3. *For any regular cardinal $\kappa > \overset{\bullet}{\uparrow}$,*

$$\binom{\kappa}{\omega_1} \longrightarrow \binom{\kappa}{\omega}_2^{1,1}.$$

Proof. Suppose $c : \kappa \times \omega_1 \rightarrow 2$ is any colouring. For every $\alpha < \kappa$, $i \in \{0, 1\}$, let S_α^i be the set $\{\beta < \omega_1 \mid c(\alpha, \beta) = i\}$. For each $\alpha < \kappa$, there is $i(\alpha)$ so that $|S_\alpha^{i(\alpha)}| = \aleph_1$, so there are $B_0 \in [\kappa]^\kappa$ and $i^* \in \{0, 1\}$ so that if $\alpha \in B_0$, then $i(\alpha) = i^*$.

Let $\langle d_\alpha : \alpha < \overset{\bullet}{\uparrow} \rangle$ be a sequence witnessing $\overset{\bullet}{\uparrow}$. For every $\alpha \in B_0$, choose $\beta(\alpha) < \omega_1$ so that $d_\beta \subseteq S_\alpha^{i^*}$. Since $\kappa > \overset{\bullet}{\uparrow}$ is regular, there is a $B_1 \in [B_0]^\kappa$ and $\beta^* < \overset{\bullet}{\uparrow}$ so that $d_{\beta^*} \subseteq S_\alpha^{i^*}$ for all $\alpha \in B_1$, and therefore $c \upharpoonright (B_1 \times d_{\beta^*})$ is constant. \square

4. PARTITION RELATIONS OPPOSITE THE UNBOUNDING NUMBER AND THE STICK-PRINCIPLE

Takahashi proved in [987Ta] that one can derive $\omega_1 \omega \not\rightarrow (\omega_1 \omega, 3)^2$ in the system $\text{ZFC} + \mathfrak{d} = \aleph_1 + \overset{\bullet}{\uparrow}$. The same can be done in the system $\text{ZFC} + \mathfrak{b} = \aleph_1 + \overset{\bullet}{\uparrow}$ as shown in [017LW]. We now show that the latter system is also sufficient to derive the negative partition relation shown to follow from CH by Baumgartner and Hajnal in [987BH].

Theorem 4.1 ([017LW]). *Suppose that κ is regular and $\lambda = \kappa^+ = \mathfrak{b}_\kappa = \mathfrak{t}(\kappa)$. Then $\lambda\kappa \not\rightarrow (\lambda\kappa, 3)^2$.*

Corollary 4.2. *If κ is regular, $\lambda = \kappa^+ = \mathfrak{b}_\kappa = \mathfrak{t}(\kappa)$, then $\alpha \not\rightarrow (\lambda\kappa, 3)^2$ for all $\alpha < \lambda^2$.*

Theorem 4.3. *Suppose that κ is an infinite regular cardinal and $\lambda = \kappa^+ = \mathfrak{b}_\kappa = \mathfrak{t}(\kappa)$. Then $\lambda^2 \not\rightarrow (\lambda\kappa, 4)^2$.*

Proof. As in the proof of Theorem 2.9, we begin by fixing

$$(37) \quad \text{A bijection } b_\xi : \xi \longleftrightarrow \kappa,$$

$$(38) \quad \text{an increasing function } u_\xi : \kappa \longrightarrow \kappa,$$

$$(39) \quad \text{a set } s_\xi \in [\lambda]^\kappa.$$

More specifically we do this in a way such that

$$(40) \quad \{u_\xi \mid \xi < \lambda\} \text{ is an unbounded family of functions in } {}^\kappa\kappa \text{ such that}$$

$$(41) \quad \forall \xi < \lambda \forall \nu < \xi (u_\nu \leq^* u_\xi),$$

$$(42) \quad \{s_\xi \mid \xi < \lambda\} \text{ witnesses } \mathfrak{t}(\kappa), \text{ i.e. } \forall X \in [\lambda]^\lambda \exists \xi < \lambda (s_\xi \subseteq X).$$

Again, as in the proof of Theorem 2.9, we are going to define a graph $\langle \lambda^2, \Psi \rangle$. For ordinals $\alpha, \beta, \gamma, \delta < \lambda$ we will have $\{\lambda\alpha + \beta, \lambda\gamma + \delta\} \in \Psi$ if and only if $\lambda\alpha + \beta \in \Lambda_{\delta, \gamma}$ or $\lambda\gamma + \delta \in \Lambda_{\beta, \alpha}$. Moreover $\lambda\alpha + \beta \in \Lambda_{\delta, \gamma}$ implies $\gamma < \alpha < \beta < \delta$. We also, for every $\gamma \in \beta \setminus (\alpha + 1)$, let $\Delta_{\beta, \alpha, \gamma} := \{\delta \in \beta \setminus \alpha \mid \lambda\gamma + \delta \in \Lambda_{\beta, \alpha}\}$. In other words, we have

$$(43) \quad \Psi = \bigcup_{\beta < \lambda} \{ \{\lambda\alpha + \beta\} \cup \{\xi\} \mid \alpha < \beta \wedge \xi \in \Lambda_{\beta, \alpha} \}$$

$$(44) \quad \text{and } \forall \beta < \lambda \forall \alpha < \beta : \Lambda_{\beta, \alpha} = \bigcup_{\gamma < \beta} \{ \lambda\gamma + \xi \mid \xi \in \Delta_{\beta, \alpha, \gamma} \}.$$

The analogy continues in that Ψ is defined in a four-layered induction. The first layer has length λ and in step $\beta < \lambda$ we define $\Lambda_{\beta, \alpha}$ for every $\alpha < \beta$. The second layer has length β and in step $\alpha < \beta$ we define $\Lambda_{\beta, \alpha}$. The third layer has length κ ,

determining in step ζ the sets $\Lambda_{\beta,\alpha,\gamma}$ for all $\gamma \in \Gamma_{\beta,\alpha,\zeta}$ where $\Gamma_{\beta,\alpha,\zeta} \in [b_{\beta}^{-1}(\zeta) + 1 \setminus (\beta + 1)]^{<\kappa}$. If $b_{\beta}(\zeta) \in \bigcup_{\xi < \zeta} \Gamma_{\beta,\alpha,\xi}$, then $\Gamma_{\beta,\alpha,\zeta} := 0$, otherwise we define the sequence $\langle \Phi'_{\beta,\alpha,\zeta,n} \mid n < \omega \rangle$ inductively by $\Phi'_{\beta,\alpha,\zeta,0} := \{b_{\beta}(\zeta)\}$ and

$$(45) \quad \Phi'_{\beta,\alpha,\zeta,n} := \left\{ \gamma \in \beta \setminus (\alpha + 1) \mid \exists \mu \in \bigcup_{\xi < \zeta} \Gamma_{\beta,\alpha,\xi} \cup \bigcup_{k < n} \Phi'_{\beta,\alpha,\zeta,k} \setminus \gamma (b_{\mu}(\gamma) < b_{\mu}(\alpha)) \right.$$

$$(46) \quad \left. \vee \exists \rho \in \bigcup_{\xi < \zeta} \Gamma_{\beta,\alpha,\xi} \cup \bigcup_{k < n} \Phi'_{\beta,\alpha,\zeta,k} \cap \mu \setminus \gamma (b_{\mu}(\gamma) < b_{\mu}(\rho)) \right\},$$

moreover we set

$$(47) \quad \Phi_{\beta,\alpha,\zeta} := \bigcup_{n < \omega} \Phi'_{\beta,\alpha,\zeta,n}, \text{ and}$$

$$(48) \quad \Gamma_{\beta,\alpha,\zeta} := \Phi_{\beta,\alpha,\zeta} \setminus \bigcup_{\xi < \zeta} \Gamma_{\beta,\alpha,\xi}.$$

Note that

$$(49) \quad |\Phi_{\beta,\alpha,\zeta}| < \kappa.$$

First of all, it is easy to prove by induction that for all $n < \omega$ the set $\Phi'_{\beta,\alpha,\zeta,n}$ has cardinality less than κ . It is comparably easy to see that the natural numbers k for which $\Phi'_{\beta,\alpha,\zeta,k}$ is nonempty, form an initial segment of ω . If κ is uncountable, then, as κ is regular, clearly $|\Phi_{\beta,\alpha,\zeta}| < \kappa$. So assume towards a contradiction that (49) fails. We have $\kappa = \omega$ and for every natural number k the set $\Phi'_{\beta,\alpha,\zeta,k}$ is finite. But then, by (45), the sequence $\langle \max(\Phi'_{\beta,\alpha,\zeta,k+1} \setminus \bigcup_{m \leq k} \Phi'_{\beta,\alpha,\zeta,m}) \mid k < \omega \rangle$ is a descending one of ordinals, a contradiction. Therefore (49).

Now let $\langle \gamma_{\beta,\alpha,\zeta,\nu} \mid \nu < \vartheta_{\beta,\alpha,\zeta} \rangle$ be the increasing enumeration of $\Gamma_{\beta,\alpha,\zeta}$. Now inductively for every $\nu < \vartheta_{\beta,\alpha,\zeta}$, let

$$(50) \quad \Xi_{\beta,\alpha,\zeta,\nu} := \bigcup \left\{ \Delta_{\delta,\gamma_{\beta,\alpha,\xi,\mu},\gamma_{\beta,\alpha,\zeta,\nu}} \mid ((\xi < \zeta \wedge \mu < \vartheta_{\beta,\alpha,\xi}) \vee (\xi = \zeta \wedge \mu < \nu)) \wedge \gamma_{\beta,\alpha,\xi,\mu} < \gamma_{\beta,\alpha,\zeta,\nu} \wedge \delta \in \Delta_{\beta,\alpha,\gamma_{\beta,\alpha,\xi,\mu}} \right\}$$

and

$$(51) \quad \Delta_{\beta,\alpha,\gamma_{\beta,\alpha,\zeta,\nu}} := \left\{ \min(s_{\xi} \setminus \Xi_{\beta,\alpha,\zeta,\nu}) \mid \xi < \beta \wedge s_{\xi} \subseteq \beta \wedge \exists \rho \leq \zeta (b_{b_{\beta}^{-1}(\rho)}(\xi) < u_{\beta}(b_{b_{\beta}^{-1}(\rho)}(\gamma_{\beta,\alpha,\zeta,\nu}))) \right\}.$$

This finishes the induction. Now we set $\Lambda_{\beta,\alpha} := \{\lambda\gamma_{\beta,\alpha,\zeta,\nu} + \delta \mid \zeta < \kappa \wedge \nu < \vartheta_\zeta \wedge \delta \in \Delta_{\beta,\alpha,\gamma_{\beta,\alpha,\zeta,\nu}}\}$ which finishes the definition of Ψ .

Now we are going to check that this provides what was demanded. So let $Q \in [\lambda^2]^4$. Let $Q = \{\lambda\alpha_k + \beta_k \mid k \in 5 \setminus 1\}$. Without loss of generality we have $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. Assume towards a contradiction, that $[Q]^2 \subseteq \Psi$. It follows that

$$(52) \quad \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \beta_4 < \beta_3 < \beta_2 < \beta_1 \text{ and } \{\lambda\alpha_k + \beta_k \mid k \in 5 \setminus 2\} \in [\Lambda_{\beta_1,\alpha_1}]^3.$$

For $i \in \{1, 2\}$ and $j \in 5 \setminus (i+1)$, let ζ_{ij} be the induction step in which $\lambda\alpha_j + \beta_j$ was added to $\Lambda_{\beta_i,\alpha_i}$ and let $\nu_{ij} < \vartheta_{\zeta_{ij}}$ be such that $\alpha_j = \gamma_{\beta_i,\alpha_i,\zeta_{ij},\nu_{ij}}$. Now we distinguish two cases:

First assume that there is an $i \in \{1, 2\}$ and a $k \in 4 \setminus (i+1)$ such that $\zeta_{ik} \leq \zeta_{i(k+1)}$. Then, either $\zeta_{ik} < \zeta_{i(k+1)}$ or both $\zeta_{ik} = \zeta_{i(k+1)}$ and $\nu_{ik} < \nu_{i(k+1)}$. As $\{\lambda\alpha_i + \beta_i, \lambda\alpha_k + \beta_k\} \in \Psi$ and $\alpha_i < \alpha_k$, we have $\lambda\alpha_k + \beta_k \in \Lambda_{\beta_i,\alpha_i}$. As $\beta_k = \gamma_{\beta_i,\alpha_i,\zeta_{ik},\nu_{ik}}$, this implies $\beta_k \in \Delta_{\beta_i,\alpha_i,\alpha_k}$. Since $\alpha_k = \gamma_{\beta_i,\alpha_i,\zeta_{ik},\nu_{ik}} < \gamma_{\beta_i,\alpha_i,\zeta_{i(k+1)},\nu_{i(k+1)}} = \alpha_{k+1}$, (50) implies

$$(53) \quad \Delta_{\beta_k,\alpha_k,\alpha_{k+1}} \subseteq \Xi_{\alpha_i,\beta_i,\zeta_{i(k+1)},\nu_{i(k+1)}}.$$

$$(54) \quad \text{As } \{\lambda\alpha_k + \beta_k, \lambda\alpha_{k+1} + \beta_{k+1}\} \in \Psi, \text{ we have } \lambda\alpha_{k+1} + \beta_{k+1} \in \Lambda_{\beta_k,\alpha_k}.$$

$$(55) \quad \text{Therefore, } \beta_{k+1} \in \Delta_{\beta_k,\alpha_k,\alpha_{k+1}}.$$

So $\beta_{k+1} \in \Xi_{\alpha_i,\beta_i,\zeta_{i(k+1)},\nu_{i(k+1)}}$ and by (50), we get $\beta_{k+1} \notin \Delta_{\beta_i,\alpha_i,\alpha_{k+1}}$. This, however, implies $\lambda\alpha_{k+1} + \beta_{k+1} \notin \Lambda_{\beta_i,\alpha_i}$ which means that $\{\lambda\alpha_i + \beta_i, \lambda\alpha_{k+1} + \beta_{k+1}\} \in [Q]^2 \setminus \Psi$, a contradiction.

Now assume that this is not the case, i.e. $\zeta_4 < \zeta_3 < \zeta_2$ and $\zeta_8 < \zeta_6$. Let ℓ be the least natural number for which $\alpha_3 \in \Phi'_{\beta_1,\alpha_1,\zeta_3,\ell}$. Then,

$$(56) \quad \alpha_2 \notin \bigcup_{\zeta < \zeta_2} \Gamma_{\beta_1,\alpha_1,\zeta} \supseteq \bigcup_{\zeta \leq \zeta_3} \Gamma_{\beta_1,\alpha_1,\zeta} \supseteq \Phi_{\beta_1,\alpha_1,\zeta_3} \supseteq \Phi'_{\beta_1,\alpha_1,\zeta_3,\ell+1},$$

so by (45),

$$(57) \quad \forall \mu \in \bigcup_{\xi < \zeta_3} \Gamma_{\beta_1,\alpha_1,\xi} \cup \bigcup_{k \leq \ell} \Phi'_{\beta_1,\alpha_1,\zeta_3,k} \setminus \alpha_2 (b_\mu(\alpha_1) < b_\mu(\alpha_2) \wedge (\forall \rho \in \bigcup_{\xi < \zeta_3} \Gamma_{\beta_1,\alpha_1,\xi} \cup \bigcup_{k \leq \ell} \Phi'_{\beta_1,\alpha_1,\zeta_3,k} \cap \mu \setminus \alpha_2 (b_\mu(\rho) \leq b_\mu(\alpha_2))).$$

As b_{α_4} is injective, $\alpha_3 \in \Gamma_{\beta_1, \alpha_1, \zeta_4}$ and $\alpha_3 \in \Phi'_{\beta_1, \alpha_1, \zeta_3, \ell}$, we have

$$(58) \quad b_{\alpha_4}(\alpha_3) < b_{\alpha_4}(\alpha_2).$$

Now let m be the least natural number such that $\alpha_4 \in \Phi'_{\beta_2, \alpha_2, \zeta_8, m}$. Then,

$$(59) \quad \alpha_3 \notin \bigcup_{\zeta < \zeta_6} \Gamma_{\beta_2, \alpha_2, \zeta} \supseteq \bigcup_{\zeta \leq \zeta_8} \Gamma_{\beta_2, \alpha_2, \zeta} \supseteq \Phi_{\beta_2, \alpha_2, \zeta_8} \supseteq \Phi'_{\beta_2, \alpha_2, \zeta_8, m+1},$$

so by (45),

$$(60) \quad \forall \mu \in \bigcup_{\xi < \zeta_8} \Gamma_{\beta_2, \alpha_2, \xi} \cup \bigcup_{k \leq m} \Phi'_{\beta_2, \alpha_2, \zeta_8, k} \setminus \alpha_3 (b_\mu(\alpha_2) < b_\mu(\alpha_3) \wedge (\forall \rho \in \bigcup_{\xi < \zeta_8} \Gamma_{\beta_2, \alpha_2, \xi} \cup \bigcup_{k \leq m} \Phi'_{\beta_2, \alpha_2, \zeta_8, k} \cap \mu \setminus \alpha_3 (b_\mu(\rho) \leq b_\mu(\alpha_3))).$$

As $\alpha_4 \in \Phi'_{\beta_1, \alpha_1, \zeta_8, m} \setminus \alpha_3$, we get $b_{\alpha_4}(\alpha_2) < b_{\alpha_4}(\alpha_3)$, contradicting (58).

Now let $H \in [\lambda^2]^{\lambda^\kappa}$. We define

$$(61) \quad A := \{\mu < \lambda \mid |H \cap \lambda(\mu + 1) \setminus \lambda\mu| = \lambda\}.$$

Let $\alpha := \min(A)$ and $\xi := \sup(A)$ and $\tau := \min\{\mu \in \lambda \setminus \xi \mid \forall \nu \in A \exists \rho < \mu \forall \sigma \in s_\rho : \lambda\nu + \sigma \in H\}$. Consider the functions

$$(62) \quad f : \kappa \longrightarrow \kappa,$$

$$(63) \quad \mu \longmapsto \min\{\nu \in \kappa \setminus \mu \mid b_\tau^{-1}(\nu) \in A\} \text{ and}$$

$$(64) \quad g : \kappa \longrightarrow \kappa,$$

$$(65) \quad \mu \longmapsto \min\{\nu < \kappa \mid \forall \rho \leq \mu (b_\tau^{-1}(\rho) \in A \rightarrow \exists \sigma < \nu \forall \varphi \in s_{b_\tau^{-1}(\sigma)} : \lambda b_\tau^{-1}(\rho) + \varphi \in H)\}.$$

Now set $h := g \circ f$ and let $\beta \in \lambda \setminus \tau$ be such that $\lambda\alpha + \beta \in H$ and u_β is unbounded over h . Let $\iota := b_\beta(\tau)$ and $\psi \in \kappa \setminus \max(\iota, b_\beta(\alpha) + 1)$ be such that $u_\beta(\psi) > h(\psi)$ and let $\gamma := b_\tau^{-1}(\min\{\mu \in \kappa \setminus \psi \mid b_\tau(\mu) \in A\})$. By (62), we have $f(\psi) \geq b_\tau(\gamma)$. Now by (64), there is a $\sigma < h(\psi)$ such that for all $\varphi \in s_{b_\tau^{-1}(\sigma)}$ we have $\lambda\gamma + \varphi \in H$. Let $\zeta < \kappa$ and $\nu < \vartheta_{\beta, \alpha, \zeta}$ be such that $\gamma = \gamma_{\beta, \alpha, \zeta, \nu}$. Then, as $|\Xi_{\beta, \alpha, \zeta, \nu}| < \kappa$, there is a $\delta \in s_{b_\tau^{-1}(\sigma)} \setminus \Xi_{\beta, \alpha, \zeta, \nu} \subseteq \Delta_{\beta, \alpha, \gamma}$. Then $\lambda\gamma + \delta \in \Lambda_{\beta, \alpha}$ and, consequently, $\{\lambda\alpha + \beta, \lambda\gamma + \delta\} \in \Psi$. \square

Corollary 4.4. *If κ is regular and $\lambda = \kappa^+ = \mathfrak{b}_\kappa = \mathfrak{I}(\kappa)$, then $\alpha \not\rightarrow (\lambda\kappa, 4)^2$ for all $\alpha < \lambda^2\kappa$.*

Corollary 4.5. *If $\mathfrak{b} = \aleph_1 = \mathfrak{I}$ then $\alpha \not\rightarrow (\omega_1\omega, 4)^2$ for all $\alpha < \omega_1^2\omega$.*

5. BAIRE CATEGORY AND A COLOURING OF SHELAH

Answering a question of Erdős and Hajnal, Shelah [975Sh] proved that CH implies that there is a colouring proving $\omega_1 \not\rightarrow [\omega_1]_\omega^2$ with no triangle with edges of three different colours. Using a similar colouring, we reduce this hypothesis to the existence of a Luzin set, i.e., an uncountable subset of ${}^\omega 2$ so that each of its uncountable subsets is nonmeagre.

Abstractly, some Luzin-type properties are relevant to proving negative partition relations, since they capture properties of sets that are inherited by all large subsets. In fact, this is the reason that assumptions like $\mathfrak{b} = \aleph_1$ or the existence of a scale are so prominent in the theorems in this paper.

Let us place Luzin sets in the picture of cardinal characteristics. The existence of a Luzin set follows from CH, cf. [914Lu] (and in fact from $\text{cof}(\mathcal{M}) = \aleph_1$). Judah and Shelah proved that it is consistent that $\text{non}(\mathcal{M}) = \aleph_1$ and there are no Luzin sets, cf. [994JS]. The existence of a Luzin set does not entail $\mathfrak{d} = \aleph_1$: starting from a model of CH, add \aleph_2 -many Cohen reals. Each uncountable subset of the Cohen reals is nonmeagre, since it cannot be a subset of any Borel meagre set (which lives in an intermediate extension of just countably many of the Cohen factors). However, $\mathfrak{d} = \aleph_2$ in this model.

Theorem 5.1. *Suppose there exists a Luzin set. Then there is a colouring proving $\omega_1 \not\rightarrow [\omega_1]_\omega^2$ with no triangle with edges of three different colours.*

Proof. Fix a Luzin set $\langle f_\alpha \mid \alpha < \omega_1 \rangle$ of size \aleph_1 . For $\alpha < \beta < \omega_1$, define $\Delta(\alpha, \beta)$ to be the least $n < \omega$ such that $f_\alpha(n) \neq f_\beta(n)$.

Let $s : \omega \rightarrow \omega$ be a function so that for every $i < \omega$, $s^{-1}[i]$ is infinite. Define the colouring $c : [\omega_1]^2 \rightarrow \omega$ to be $c(\alpha, \beta) := s(\Delta(\alpha, \beta))$.

There are no triangles which get three colours from c .

Claim 5.2. For distinct $\alpha, \beta, \gamma < \omega_1$, the smaller two among $\Delta(\alpha, \beta), \Delta(\beta, \gamma), \Delta(\gamma, \alpha)$ are equal.

Proof of Claim. Towards a contradiction, suppose $\Delta(\alpha, \gamma) < \Delta(\beta, \gamma) \leq D(\alpha, \beta)$. By definition of Δ ,

$$f_\alpha(\Delta(\alpha, \gamma)) \neq f_\gamma(\Delta(\alpha, \gamma)),$$

but since $\Delta(\alpha, \gamma) < \Delta(\beta, \gamma)$,

$$f_\gamma(\Delta(\alpha, \gamma)) = f_\beta(\Delta(\alpha, \gamma)).$$

In total,

$$f_\alpha(\Delta(\alpha, \gamma)) \neq f_\beta(\Delta(\alpha, \gamma))$$

but this contradicts $\Delta(\alpha, \gamma) < D(\alpha, \beta)$. ©

Now we show that c satisfies $\omega_1 \not\rightarrow [\omega_1]_\omega^2$.

Claim 5.3. If $X \subseteq {}^\omega 2$ is nonmeagre, then for all but finitely many n , there exists some $t \in {}^n 2$ and $f, g \in X$ so that $t \frown \langle 0 \rangle \subseteq f$ and $t \frown \langle 0 \rangle \subseteq g$.

Proof of Claim. Suppose not. Then there are infinitely many n so that for every $f \in X$, $f(n) = i_{f \upharpoonright n}$ for some $i_{f \upharpoonright n}$ which depends only on $f \upharpoonright n$. The closure $\text{cl}(X)$ then must also have this property, which implies that the interior of $\text{cl}(X)$ is empty, and hence X is nowhere dense. ©

Suppose $A \subseteq \omega_1$. Then by the Luzin property, $\{f_\alpha \mid \alpha \in A\}$ is nonmeagre, so there is some n^* large enough so that all $n \geq n^*$ satisfy the condition in the last claim. Let $m < \omega$ be arbitrary. Choose $n \geq n^*$ so that $s(n) = m$ and $\alpha, \beta \in A$ so that $f_\alpha \upharpoonright n = f_\beta \upharpoonright n$ but $f_\alpha(n) = 0$ and $f_\beta(n) = 1$. Then $\Delta(\alpha, \beta) = m$, and $c(\alpha, \beta) = s(\Delta(\alpha, \beta)) = m$. □

Corollary 5.4. *If $\text{cof}(\mathcal{M}) = \aleph_1$, then there is a colouring proving $\omega_1 \not\rightarrow [\omega_1]_\omega^2$ with no triangle with edges of three different colours.*

Proof. It suffices to show that there is a Luzin set under these hypotheses. We give the argument of this classical fact here.

Let $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ be a cofinal family of meagre sets. Let $B_\alpha = \bigcup_{\beta \leq \alpha} A_\beta$, so $\langle B_\alpha \mid \alpha < \omega_1 \rangle$ is a \subseteq -increasing cofinal family of meagre sets. Thin out so that $B_\alpha \neq B_\beta$ if $\alpha \neq \beta$. For each $\alpha < \omega_1$, pick $x_\alpha \in B_{\alpha+1} \setminus B_\alpha$. Then $\{x_\alpha \mid \alpha < \omega_1\}$ is a Luzin set: none of its uncountable subsets are contained in any single B_α . □

6. QUESTIONS

This paper leaves open many natural questions. On the relationship between cardinal invariants and \mathfrak{b} , even the following is not known:

Question 6.1. Does $\dot{\mathfrak{p}} = \aleph_1$ imply $\mathfrak{b} = \aleph_1$?

We suspect that the assumption $\mathfrak{d} = \aleph_1$ appearing in the theorems above and in Larson [998La] can be reduced somewhat. As a starting point, we ask:

Question 6.2. Does $\dot{\mathfrak{p}} = \aleph_1$ imply $\omega_1\omega^2 \not\rightarrow (\omega_1\omega^2, 3)^2$ or even $\omega_1\omega \not\rightarrow (\omega_1\omega, 3)^2$? What about $\mathfrak{b} = \aleph_1$?

Question 6.3. Does $\dot{\mathfrak{p}} = \aleph_1$ imply $\omega_1^2 \not\rightarrow (\omega_1\omega, 4)^2$? What about $\mathfrak{b} = \omega_1$?

The positive partition relations are known to follow from $\text{MA} + \neg \text{CH}$ (cf. Section 1) and can be forced with by iterating c.c.c. posets which add homogeneous sets to counterexamples using finite conditions. However, these posets destroy $\dot{\mathfrak{p}} = \aleph_1$ and $\mathfrak{b} = \aleph_1$.

Perhaps easier would be to consider the polarised partition relation, where our basic question is:

Question 6.4. Does $\dot{\mathfrak{p}} = \aleph_1$ imply $(\omega_1) \not\rightarrow (\omega_1)_2^{1,1}$?

We would also be interested in any relationships of these relations with the category invariants.

Whether $\omega_1^2 \rightarrow (\omega_1^2, 3)^2$ is consistent is a long-standing open question of Baumgartner. In our context, we would be interested in reducing the hypotheses used by Larson ($\mathfrak{d} = \aleph_1$) and Weinert–Lambie–Hanson ($\mathfrak{b} = \dot{\mathfrak{p}} = \aleph_1$) to prove the negative relation. A natural mutual strengthening of these results would be:

Question 6.5. Does $\mathfrak{b} = \aleph_1$ imply $\omega_1^2 \not\rightarrow (\omega_1^2, 3)^2$?

The arguments here from cardinal characteristics roughly calibrate the strength of negative partition relations. We can ask if there is a direct implication.

Question 6.6. Does $\omega_1^2 \not\rightarrow (\omega_1\omega, 4)$ imply $\omega_1\omega \not\rightarrow (\omega_1\omega, 3)^2$?

Question 6.7. Does $\omega_1^2 \not\rightarrow (\omega_1\omega, 4) \wedge \omega_1^2 \not\rightarrow (\omega_1^2, 3)$ imply $\omega_1\omega \not\rightarrow (\omega_1\omega, 3)^2$?

In this paper, we focused on ω_1 for the value of the cardinal characteristic in the hypothesis and for the resources in the partition relations. However, it may be possible that this is not necessary.

Question 6.8. Does $\mathfrak{b} \leq \mathfrak{p}$ hold by way of ZFC? For which cardinals κ does $\lambda = \kappa^+$ and $\mathfrak{p}(\kappa) = \lambda$ imply the existence of an unbounded family of cardinality ${}^\lambda\lambda$?

Question 6.9. Is $\min(\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})) \leq \mathfrak{p}$?

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