VARIATIONS OF THE STICK PRINCIPLE

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Abstract. We solve several problems about different variants of the $\stick$ principle. First, we separate $\stick$ principles which prescribe certain order-types for the members of the family. Then, we separate a principle called Superstick from CH, answering a question of Primavesi.

1. Preliminaries

For an ordinal δ and a set X, we use the notation $[X]^\delta = \{x \subseteq X : \mathrm{ot}(x) = \delta\}$ (here $\mathrm{ot}(x)$ is the order-type of x). We are interested in the following principle:

$$(1) \qquad \stick_\delta = \min\{|X| : X \subseteq [\omega_1]^\delta \text{ and } \forall y \in [\omega_1]^{\omega_1} \exists x \in X (x \subseteq y)\}.$$

We will denote $\stick_\omega$ simply by $\stick$. Using different notation, Baumgartner [1] introduced these principles (specifically, the restrictions on order-types) in connection with partition relations.

The principle $\stick = \aleph_1$, also denoted $(\stick)$, has been the focus of much study. It is a natural weakening of Ostaszewski's club principle $\clubsuit$.

Definition 1.1. $\clubsuit$ asserts that there is a subset $\langle x_\alpha : \alpha < \omega_1\rangle$ of $[\omega_1]^\omega$ so that $x_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$, and for all $y \in [\omega_1]^{\omega_1}$ there is $\alpha < \omega_1$ so that $x_\alpha \subseteq y$.

Thilo Weinert asked whether $\stick = \stick_\alpha$ for all countable ordinals $\alpha < \omega_1$. It is not difficult to see that $\stick = \aleph_1$ implies that $\stick_{\omega^2} = \aleph_1$. In section 2, we give a negative answer, producing for any $\delta_0 < \delta_1$ separated by a multiplicatively indecomposable ordinal a model of $\stick_{\delta_0} < \stick_{\delta_1}$.

In his thesis [5], Alexander Primavesi was interested in Juhasz's question of whether $\clubsuit$ implies the existence of a Suslin tree. Towards a positive answer, he defined the principle Superclub, a strengthening of $\clubsuit$ still implied by $\diamondsuit$. The key point is that Superclub is sufficient to construct a Suslin tree.

Definition 1.2. Superclub is the principle stating that there exists a sequence $\langle x_\delta : \delta \in \mathrm{Lim}(\omega_1)\rangle$ so that for any $y \in [\omega_1]^{\omega_1}$, there is $x \in [\omega_1]^{\omega_1}$ so that $x \subseteq y$ and $\{\delta \in \mathrm{Lim}(\omega_1) : x \cap \delta = x_\delta\}$ is stationary.

However, the question remained open as to whether Superclub was in fact simply equivalent to $\diamondsuit$ in ZFC.

Primavesi also defined a related principle, Superstick, between $\stick = \aleph_1$ and CH.

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Definition 1.3. Superstick is the principle stating that there exists a collection $X \subseteq [\omega_1]^{<\omega_1}$ of size $\aleph_1$ so that for any $y \in [\omega_1]^{\omega_1}$, there is a $\subset$-increasing sequence $\langle x_\alpha : \alpha < \omega_1 \rangle \subseteq X$ so that $x_\alpha \subseteq y$ for all $\alpha < \omega_1$.

He proved that $\clubsuit +$ Superstick implies Superclub and asked whether Superstick was just equivalent to CH.

In Section 3, we give negative answers to both questions, producing a model where Superclub holds but CH fails (and hence so does $\diamondsuit$).

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2. Consistency of $\stick_{\delta_0} < \stick_{\delta_1}$

In the introduction, we observed that $\stick = \aleph_1$ implies that $\stick_{\omega^2} = \aleph_1$. More generally,

Observation 2.1. *Suppose $\delta_0 \leq \delta_1$ are countable ordinals. If $\stick_{\delta_1} = \aleph_1$, then also $\stick_{\delta_0} = \aleph_1$. If $\stick_{\delta_0} = \stick_{\delta_1} = \aleph_1$, then $\stick_{\delta_1 \cdot \delta_0} = \aleph_1$.*

Proof. The first claim is easy. For the second claim, let $\langle d^0_\xi : \xi < \omega_1 \rangle$ and $\langle d^1_\xi : \xi < \omega_1 \rangle$ witness $\stick_{\delta_0} = \aleph_1$ and $\stick_{\delta_1} = \aleph_1$, respectively.

Let D be the collection of members of $[\omega_1]^{\delta_1 \cdot \delta_0}$ of the form $\bigcup \{d^1_\xi : \xi \in d^0_{\xi^*}\}$ for some $\xi^* < \omega_1$. This clearly has size $\aleph_1$. For any $y \in [\omega_1]^{\omega_1}$, define inductively using the $\stick_{\delta_1}$ property an increasing sequence of ordinals $\langle \xi_i : i < \omega_1 \rangle$ such that for all $i < \omega_1$, $d^1_{\xi_i} \subseteq y \setminus \sup_{j<i} d^1_{\xi_j}$. Then pick ξ^* so that $d^0_{\xi^*} \subseteq \{\xi_i : i < \omega_1\}$. The set $\bigcup \{d^1_\xi : \xi \in d^0_{\xi^*}\}$ is in D and is a subset of y. □

In fact, we will show that these are the only ZFC implications between the principles $\stick_\delta = \aleph_1$ for $\delta < \omega_1$.

First, let us give a quick application of the ideas here to solve a problem from Geoff Galgon's thesis [4]. There he considered the principle ($\stick^{\mathrm{ad}}$), that $\stick = \aleph_1$ is witnessed by an almost disjoint family.

Definition 2.2. ($\stick^{\mathrm{ad}}$) is the principle stating that there exists $X \subseteq [\omega_1]^\omega$ so that

- for any $x \neq x'$ from X, $x \cap x'$ is finite,
- for all $y \in [\omega_1]^{\omega_1}$, there is $x \in X$ so that $x \subseteq y$.

He observed that ($\stick^{\mathrm{ad}}$) is a weakening of $\clubsuit$ which suffices to prove that there is a size $\aleph_1$ family of functions $\omega_1 \to \omega_1$ which is maximal with respect to the property of being almost disjoint. He asked whether $\stick = \aleph_1$ is equivalent to ($\stick^{\mathrm{ad}}$), and proved that it is under the assumption $\neg$CH. We prove it without any assumptions.

Proposition 2.3. ($\stick^{\mathrm{ad}}$) *is equivalent to* $\stick = \aleph_1$.

Proof. If $\stick = \aleph_1$, then $\stick_{\omega^2} = \aleph_1$, so let $\langle z_\xi : \xi < \omega_1 \rangle$ be a sequence witnessing $\stick_{\omega^2} = \aleph_1$. We will build an almost disjoint family $\langle x_\xi : \xi < \omega_1 \rangle$ of subsets of ω_1 of order-type ω satisfying $x_\xi \subseteq z_\xi$.

The construction is by induction on ξ. At stage ξ, let $e_\xi : \xi \to \omega$ be an injection. Build x_ξ in ω stages, so that $x_\xi = \{a^n_\xi : n < \omega\}$.

At step n, choose a^n_ξ in $z_\xi \setminus \bigcup\{x_\zeta : e_\xi(\zeta) < n\}$ above $\sup\{a^m_\xi : m < n\}$. This is possible since z_ξ has order-type ω^2 and $\bigcup\{x_\zeta : e_\xi(\zeta) < n\}$ has order-type $< \omega^2$.

For any $y \in [\omega_1]^\omega$, there is some ξ with $z_\xi \subseteq y$. Then $x_\xi \subseteq z_\xi \subseteq y$. □

Remark 2.4. Given $\stick_\alpha = \aleph_1$, the construction can be modified to produce almost disjoint families witnessing $\stick_\alpha = \aleph_1$ for any $\alpha < \omega_1$.

Now we proceed to the main result of this section.

Theorem 2.5. *Suppose* CH *holds. Suppose $\delta_0 < \delta_1$ and δ_1 is of the form ω^{ω^ϵ} for some ordinal $\epsilon < \omega_1$ (i.e., multiplicatively indecomposable). Then there is a cardinal preserving poset which forces $\stick_{\delta_0} = \aleph_1$ and $\stick_{\delta_1} = \aleph_2$.*

Proof. The forcing poset follows closely a technique of Dzamonja and Shelah [2], with the additional ingredient of a forcing of Baumgartner [1].

Suppose CH holds in the ground model. Let $\mathcal{A} = \langle A_\alpha : \alpha < \omega_2\rangle$ be a family of $\aleph_2$ subsets of ω_1 whose pairwise intersections are countable. Let $\langle N_\xi : \xi < \omega_1\rangle$ be a continuous $\subseteq$-increasing sequence of countable elementary submodels of $(H(\chi); \in, \prec, \mathcal{A})$ (here χ is a regular cardinal sufficiently large to contain $\mathcal{A}$ and $\prec$ is a well-order of $H(\chi)$) so that $\langle N_\zeta : \zeta \leq \xi\rangle \in N_{\xi+1}$ for all $\xi < \omega_1$ and $\gamma_\xi := N_\xi \cap \omega_1 \in \omega_1$.

Let $\mathbb{P}$ be the poset whose conditions are partial functions $p : \omega_2 \to [\omega_1]^{<\delta_1}$ with countable domain so that for all $\alpha \in \mathrm{dom}(p)$,

(1) $f(\alpha) \subseteq A_\alpha$,
(2) (respects submodels) $f(\alpha) \cap \gamma_\xi \in N_{\xi+1}$ for all $\xi < \omega_1$.

The ordering is defined so that $p \leq q$ exactly when

(1) $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and for all $\alpha \in \mathrm{dom}(q)$, $p(\alpha) \supseteq q(\alpha)$.
(2) $\mathrm{diff}(p, q) := \{\alpha \in \mathrm{dom}(p) \cap \mathrm{dom}(q) : p(\alpha) \neq q(\alpha)\}$ is finite.
(3) for every $\alpha < \beta$ in the domain of q, $p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta)$.

We will also use an auxiliary ordering, defined as $p \leq^* q$ if and only if $p \leq q$ and for all $\alpha \in \mathrm{dom}(q)$, $p(\alpha) = q(\alpha)$. Notice that $(\mathbb{P}, \leq^*)$ is countably closed: the union of a countable $\leq^*$-decreasing sequence of conditions is itself a condition.

Claim 2.6. *$\mathbb{P}$ preserves ω_1.*

Proof. Let G be generic for $\mathbb{P}$ over V. Suppose in $V[G]$ there is an injection $f : (\omega_1)^V \to \omega$. Then there is some $p \in \mathbb{P}$ which forces this. We inductively define sequences of conditions $\langle p_i : i < \omega_1\rangle$, $\langle q_i : i < \omega_1\rangle$, and a sequence of natural numbers $\langle n_i : i < \omega_1\rangle$ so that:

(1) $p_0 \leq p$, and $\langle p_i : i < \omega_1\rangle$ is $\leq^*$-decreasing.
(2) $q_i \leq p_i, \mathrm{dom}(q_i) = \mathrm{dom}(p_i)$ and $q_i \Vdash n_i = \dot{f}(i)$.
(3) If $p_i(\alpha) \neq q_i(\alpha)$ then $\alpha \in \mathrm{dom}(p_j)$ for some $j < i$.

The construction is straightforward: at each stage i, let q_i be an extension of $\bigcup_{j<i} p_j$ (which is a condition since $\langle p_j : j < i\rangle$ is $\leq^*$-decreasing) that forces a value for $\dot{f}(i)$, and let

$$p_i(\alpha) = \bigcup_{j<i} p_j \cup q_i \restriction (\omega_2 \setminus \bigcup_{j<i} \mathrm{dom}(p_j)).$$

The sequence $\langle \bigcup_{j<i} \mathrm{dom}(p_j) : i < \omega_1\rangle$ is continuous. By the construction, $\mathrm{diff}(p_i, q_i) \subseteq \bigcup_{j<i} \mathrm{dom}(p_j)$. By Fodor's lemma, there are $Y \subseteq \omega_1$ uncountable and $d^* \subseteq \omega_2$ finite so that $\mathrm{diff}(p_i, q_i) = d^*$ for all $i \in Y$.

For all distinct ordinals $\alpha, \beta \in d^*$, there are only countably many options for the value of $p(\alpha) \cap p(\beta)$ for $p \in \mathbb{P}$. This is because $A_\alpha \cap A_\beta$ is countable, so there is $\xi < \omega_1$ with $A_\alpha \cap A_\beta \subseteq \gamma_\xi$. Now $p(\alpha) \cap p(\beta) = (p(\alpha) \cap \gamma_\xi) \cap (p(\beta) \cap \gamma_\xi) \in N_{\xi+1}$.

Thin Y to an uncountable subset Y' so that the value of each of these intersections is fixed. Then $\langle q_i : i \in Y' \rangle$ are pairwise compatible and therefore $\langle n_i : i \in Y' \rangle$ are pairwise distinct since p forces that f is an injection. But this is impossible since Y' is uncountable. □

Claim 2.7. *$\mathbb{P}$ is $\aleph_2$-c.c., and hence cardinal preserving.*

Proof. A straightforward Δ-system argument, using CH. □

Claim 2.8. *In the extension $V[G]$, $⫯_{\delta_1} > \aleph_1$.*

Proof. The generic function $B : \alpha \mapsto \bigcup\{p(\alpha) : p \in G\}$ maps $\omega_2 \to P(\omega_1)$ and can be thought of as a sequence of subsets $\langle B(\alpha) : \alpha < \omega_2 \rangle$. For each $\alpha < \omega_2$ and $\xi < \omega_1$, there is a dense set D^ξ_α of $p \in \mathbb{P}$ so that $p(\alpha) \setminus \xi \neq \emptyset$, since for an arbitrary $q \in \mathbb{P}$ we may add an ordinal to $q(\alpha)$ above the supremum of $A_\alpha \cap A_\beta$ for any $\beta \in \mathrm{dom}(q) \setminus \{\alpha\}$ (and this respects submodels).

Now in $V[G]$ if $x \in [\omega_1]^{\delta_1}$, then there is at most one $\alpha < \omega_2$ so that $x \subseteq B(\alpha)$. Therefore $⫯_{\delta_1} > \aleph_1$. □

Claim 2.9. *In the extension $V[G]$, $⫯_{\delta_0} = \aleph_1$.*

We may assume δ_0 is a limit ordinal, otherwise pass to $\delta_0 + \omega$. Suppose $p \in \mathbb{P}$ forces $\dot{X}$ to be an uncountable subset of ω_1. We will find a condition below p which forces that $\dot{X}$ contains a ground model subset of order-type δ_0. As $\dot{X}$ was arbitrary, it is forced that $[\omega_1]^{\delta_0} \cap V$ witnesses $⫯_{\delta_0} = \aleph_1$.

We can find $\langle N^*_\xi : \xi \leq \delta_1 \rangle$ a continuous increasing sequence of countable submodels of $(H(\chi^*); \in, \prec, \mathbb{P}, p)$ (for $\chi^* > \chi$ sufficiently large to contain $\mathbb{P}, p$ and $\prec$ a well-order extending the $\prec$ from $H(\chi)$ to $H(\chi^*)$) so that for any ξ,

(1) $\langle N^*_\zeta : \zeta \leq \xi \rangle \in N^*_{\xi+1}$.
(2) $N^*_\xi \cap H(\omega_1) = N_{i(\xi)}$ for some $i(\xi) < \omega_1$.

For each $\xi < \delta_1$, let $\langle \psi^\xi_n : n < \omega \rangle$ be the $\prec$-least enumeration of all formulas with parameters in $N^*_{\xi+1}$.

We can easily define a sequence $\langle p_{\omega\xi+n} : \xi < \delta_1, n < \omega \rangle$ so that

(1) $\langle p_{\omega\xi+n} : \xi < \delta_1, n < \omega \rangle$ is $\leq^*$-decreasing, and continuous at limit ordinals ($p_\lambda = \bigcup_{\zeta<\lambda} p_\zeta$ for limit λ).
(2) $p_{\omega\xi+n} \in N^*_{\xi+1}$ for all ξ, n.
(3) If there is $r \leq p_{\omega\xi+n}$ for which there exists some β so that $\psi^\xi_n(r, \beta)$ holds, then for the $\prec$-least such r, $p_{\omega\xi+n+1}$ is so that $\mathrm{dom}(p_{\omega\xi+n+1}) = \mathrm{dom}(r)$ and $p_{\omega\xi+n+1}(\alpha) = r(\alpha)$ if $\alpha \in \mathrm{dom}(r) \setminus \mathrm{dom}(p_{\omega\xi+n})$. Otherwise let $p_{\omega\xi+n+1} = p_{\omega\xi+n}$.

Let $p^* = \bigcup\{p_\xi : \xi < \omega\delta_1\}$. Let $q^* \leq p^*$ be the $\prec$-least such that $q^* \Vdash \beta^* \in \dot{X}$ for some $\beta^* > N^*_{\delta_1} \cap \omega_1$. Let $u^* = \mathrm{diff}(q^*, p^*)$. There is some limit ordinal $\xi^* < \delta_1$ so that ξ^* has a cofinal subset of order-type δ_0 and $\sup(\bigcup_{\alpha \in u^*} q^*(\alpha) \cap N^*_{\xi^*} \cap \omega_1) < N^*_{\xi^*} \cap \omega_1$. Otherwise, for a tail end of $\xi < \delta_1$, $\sup(\bigcup_{\alpha \in u^*} q^*(\alpha) \cap N^*_\xi \cap \omega_1) = N^*_\xi \cap \omega_1$, but the definition of $\mathbb{P}$ specifies that $q^*(\alpha)$ has order-type less than δ_1 for any $\alpha \in u^*$.

Let $\epsilon^* = \sup(\bigcup_{\alpha\in u^*} q^*(\alpha) \cap N^*_{\xi^*} \cap \omega_1)$. For each $\alpha \in u^*$, let t_α be the order-type of $q^*(\alpha)$.

Let $\langle \xi_i : i < \delta_0\rangle$ be the $\prec$-least increasing sequence of ordinals with limit ξ^* with $\epsilon^* \in N^*_{\xi_0}$ and $\alpha, q^*(\alpha)\cap\epsilon^*, t_\alpha \in N^*_{\xi_0}$ for all $\alpha \in u^*$ (this is possible since $q^*(\alpha)\cap\epsilon^* < \gamma_{i(\xi)}$ for some $\xi_{-1} < \xi^*$, and then by the definition of $\mathbb{P}$, $q^*(\alpha) \cap \epsilon^* \in N^*_{\xi_{-1}+1}$).

We define by induction on $i < \delta_0$ conditions r_i, a natural number m_i, a formula φ_i, and ordinals $\beta_i \in N^*_{\xi_i+1}$. Let $\varphi_i(x, y)$ be the formula

(1) $x \in \mathbb{P}$ and $y > N^*_{\xi_i} \cap \omega_1$.
(2) $x \Vdash y \in \dot{X}$.
(3) For all $\alpha \in u^*$, $x(\alpha) \cap (N^*_{\xi_i} \cap \omega_1) = q^*(\alpha) \cap \epsilon^*$ and $\mathrm{ot}(x(\alpha)) = t_\alpha$.
(4) If $x(\alpha) \neq p_{\omega\xi_i+m_i}(\alpha)$ for some α in their common domain, then $\alpha \in u^*$.

Note that all parameters used in $\varphi_i(x, y)$ are from $N^*_{\xi_i+1}$.

Let m_i be the index of φ_i. Since $\varphi_i(q^*, \beta^*)$ holds and $q^* \le p_\xi$ for all ξ, take β_i to be the least ordinal for which there is some x so that $\varphi_i(x, \beta_i)$ holds, so $\beta_i \in N^*_{\xi_i+1}$. Take r_i to be the $\prec$-least so that $r_i \le p_{\omega\xi_i+m_i}$ and $\varphi_i(r_i, \beta_i)$ holds. By the construction of $\langle p_{\omega\xi+n} : \xi < \delta_1, n < \omega\rangle$, $\mathrm{dom}(r_i) = \mathrm{dom}(p_{\omega\xi_i+m_i+1})$.

Let r^* be defined on $\bigcup_{i<\delta_0} \mathrm{dom}(r_i)$ by $\alpha \mapsto \bigcup_{i<\delta_0} r_i(\alpha)$. We can check that r^* is a condition; the main points are:

- $r^*(\alpha) = p^*(\alpha)$ if $\alpha \notin u^*$,
- for $\alpha \in u^*$ the domains of the $r_i(\alpha)$ were chosen to be pairwise disjoint above $q^*(\alpha) \cap \epsilon^*$, so the union is a function,
- the order-type of $r^*(\alpha)$ is less than $t_\alpha\delta_0 < \delta_1$ for each $\alpha \in u^*$,
- for any $\xi < \omega_1$, $r^*(\alpha)\restriction\gamma_\xi \in N_{\xi+1}$. This is only nontrivial to check if ξ corresponds to a limit stage of the construction of the r_i's, and in this case it follows easily since we ensured at each stage that the construction took place inside the appropriate submodel.

Now $r^* \Vdash \beta_i \in \dot{X}$ for all $i < \delta_0$. The set $\{\beta_i : i < \delta_0\}$ is a member of V, and the β_i were chosen to be increasing by item (1) of the definition of φ_i, so $\{\beta_i : i < \delta_0\}$ has order-type δ_0. □

We conclude this section with some remarks on the situation when $\stick > \aleph_1$. In this case, it is not even clear if $\stick = \stick_{\omega^2}$: the problem is that we are unable to guess the indices of a $\stick$ family with the $\stick$ set itself. In [3], the following cardinal invariant is introduced:

Definition 2.10. $\stick'$ is the minimum $\kappa \ge \aleph_1$ so that there is $X \subseteq [\kappa]^{<\omega_1}$ such that $|X| = \kappa$ and for all $y \in [\kappa]^{\omega_1}$ there is $x \in X$ with $x \subseteq y$.

It is not difficult to see that $\stick \le \stick'$. If $\stick = \stick'$, then the argument of Observation 2.1 goes through. It remains open whether it is consistent that $\stick < \stick'$.

3. Superclub + ¬CH

Theorem 2.5 can be used to give a model where $\stick = \aleph_1$ but Superstick fails, since Superstick implies $\stick_\delta = \aleph_1$ for all $\delta < \omega_1$.

We now prove that it is consistent that Superstick (and even Superclub) holds but CH fails.

Theorem 3.1. *Suppose $\diamondsuit$ holds and there is an inaccessible cardinal κ. Then there is a poset forcing* $\mathsf{Superstick} + \clubsuit + \neg\mathsf{CH}$.

Proof. We define the poset as an iteration using a similar kind of supports as in the previous section. The forcing will add κ-many Cohen reals using the mixed support, with posets interleaved to force the ground model reals to witness $\mathsf{Superstick}$. The iteration $\mathbb{P}_\alpha$ is defined inductively with factors $\dot{\mathbb{Q}}_\beta$, where $\mathbb{Q}_\beta$ is either $\mathrm{Add}(\omega, 1)$ (the set of all finite functions $\omega \to 2$) or a name for a poset $\mathrm{Thread}(y)$ in $V^{\mathbb{P}_\beta}$ The *Cohen coordinates* are β where $\mathbb{Q}_\beta = \mathrm{Add}(\omega, 1)$, and β where $\mathbb{Q}_\beta = \mathrm{Thread}(y)$ are called *thread coordinates*. If y is an uncountable subset of ω_1 in the extension by $\mathbb{P}_\beta$, then $\mathrm{Thread}(y)$ is the poset $\{x \in [\omega_1]^{<\omega_1} \cap V : x \subseteq y\}$ defined in this extension, ordered by end-extension.

Then $\mathbb{P}_\alpha$ is the set of all countable-domain partial functions $p : \alpha \to V$ where $p(\beta)$ is forced by $p \restriction \beta \in \mathbb{P}_\beta$ to be a canonical name for an element of $\mathbb{Q}_\beta$. The ordering on $\mathbb{P}_\alpha$ is so that $p \leq q$ exactly when

(1) $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$.
(2) For all $\beta \in \mathrm{dom}(q)$, $p \restriction \beta \Vdash p(\beta) \leq q(\beta)$.
(3) $\mathrm{diff}(p, q) := \{\beta \in \mathrm{dom}(p) \cap \mathrm{dom}(q) : p \restriction \beta \Vdash p(\beta) \neq q(\beta)\}$ is finite.

We have an auxiliary order so that $p \leq^* q$ if and only if $p \leq q$ and for all $\alpha \in \mathrm{dom}(q)$, if $\mathbb{Q}_\alpha = \mathrm{Add}(\omega, 1)$ then $p(\alpha) = q(\alpha)$.

Finally, use the usual bookkeeping to arrange $\mathbb{P} := \mathbb{P}_\kappa$ be so that for any name for a uncountable set y in the final extension which appears by some initial segment of the iteration, there is $\beta < \kappa$ so that $\mathbb{Q}_\beta$ is the name of the poset $\mathrm{Thread}(y)$ (and every subset of ω_1 in the final model will appear at some initial stage).

This kind of iteration was developed by Fuchino, Soukup, and Shelah [3], who called it CS^*-iteration. The following lemma is useful for the proofs that follow.

Lemma 3.2. *Let $p, q \in \mathbb{P}$ be such that $p \leq q$. Then there is $p' \leq p$ so that for any $\alpha \in \mathrm{diff}(p', q)$, $p' \restriction \alpha$ forces a value (as a member of V) for $p'(\alpha)$.*

Proof. Let $p_0 = p$ and define inductively

$$\alpha_n = \max\{\alpha \in \mathrm{diff}(p_n, q) : p_n \restriction \alpha \text{ doesn't force a value for } p_n(\alpha)\}.$$

Extend p_n to p_{n+1} by strengthening $p_n \restriction \alpha$ to force a value for $p_n(\alpha)$. This process must terminate after finitely many steps, since the α_n's are decreasing, and it terminates in a condition p' as required by the lemma. □

Using that κ is inaccessible together with the usual Δ-system arguments easily gives

Claim 3.3. $\mathbb{P}$ *satisfies the κ-c.c.*

Similarly as in the last section, we can prove

Claim 3.4. $\mathbb{P}$ *preserves ω_1 and $\clubsuit$.*

Proof. Fix a $\diamondsuit$-sequence $\langle x_\xi : \xi \in \mathrm{Lim}(\omega_1)\rangle$, $\dot{y}$ a name for an uncountable subset of ω_1, and $\dot{f}$ a name of a function $\omega_1 \to \omega$. We will show that there is a dense set of conditions in $\mathbb{P}$ which force some $\check{x}_\xi \subseteq \dot{y}$ and $\dot{f}$ not one-to-one. We inductively define sequences of conditions $\langle p_i : i < \omega_1\rangle$, $\langle q_i : i < \omega_1\rangle$, an increasing sequence of countable ordinals $\langle \gamma_i : i < \omega_1\rangle$, and a sequence of natural numbers $\langle n_i : i < \omega_1\rangle$ so that:

(1) $p_0 = p$, and $\langle p_i : i < \omega_1\rangle$ is $\leq$-decreasing.
(2) For every $i < \omega_1$, $\langle p_j : j < i\rangle$ has a greatest lower bound $p_i^* \in \mathbb{P}$.
(3) $q_i \leq p_i^*$, $q_i \Vdash \gamma_i \in \dot{y}$, and $q_i \Vdash n_i = \dot{f}(i)$.
(4) $q_i \leq p_i$ and $\mathrm{dom}(q_i) = \mathrm{dom}(p_i)$.
(5) For all $\alpha \notin \mathrm{diff}(p_i^*, q_i)$, $p_i(\alpha) = q_i(\alpha)$.
(6) For all Cohen coordinates $\alpha \in \mathrm{diff}(p_i^*, q_i)$, $p_i(\alpha) = p_i^*(\alpha)$.
(7) For all thread coordinates $\alpha \in \mathrm{diff}(p_i^*, q_i)$, $q_i(\alpha) = \check{t}$ for some $t \in V$ and $p_i(\alpha) = q_i(\alpha)$.

Suppose we are at stage i and the construction has been done for all $j < i$.

We check that $\langle p_j : j < i\rangle$ has a lower bound. This is nontrivial only if i is a limit ordinal. Let $\alpha \in \bigcup_{j<i} \mathrm{dom}(p_j)$. If α is a coordinate for which there is $j_0 < i$ so that for all $j_0 \leq j < i$, $\alpha \notin \mathrm{diff}(p_j^*, q_j)$, then $p_j(\alpha)$ stabilizes. Otherwise, if α is a Cohen coordinate, then (6) ensures that $p_j(\alpha)$ is fixed for $j < i$ large enough so that $\alpha \in \mathrm{dom}(p_j)$. In either of these two cases, let $p_i^*(\alpha)$ be this stable value.

In the remaining case, α is a thread coordinate and there is a set J unbounded in i so that $\alpha \in \mathrm{diff}(p_j^*, q_j)$ for all $j \in J$. Then $q_j(\alpha) = \check{t}_j$ for some $t_j \in V$. Now $t^* := \bigcup_{j\in J} t_j \in V$, so we can let $p_i^*(\alpha)$ be the canonical name for t^*.

At each stage i, use Lemma 3.2 to find q_i be an extension of p_i^* satisfying (3) and (7) with γ_i larger than γ_j for every $j < i$. The rest of the construction is determined by (4)–(7).

The sequence $\langle \bigcup_{j<i} \mathrm{dom}(p_j) : i < \omega_1\rangle$ is continuous. By (5), $\mathrm{diff}(p_i, q_i) \subseteq \mathrm{diff}(p_i^*, q_i) \subseteq \bigcup_{j<i} \mathrm{dom}(p_j)$. By Fodor's lemma, there are $Y \subseteq \omega_1$ uncountable and $d^* \subseteq \omega_2$ finite so that $\mathrm{diff}(p_i, q_i) = d^*$ for all $i \in Y$. Thin Y further to Y' to get a fixed value for $\langle q_i(\alpha) : \alpha \in d^*\rangle$ for all $i \in Y'$.

The sequence $\langle \gamma_i : i \in Y\rangle$ is an uncountable subset of ω_1 in V, so there is some $\xi \in \mathrm{Lim}(\omega_1)$ with $x_\xi \subseteq \langle \gamma_i : i \in Y\rangle$.

Take j large enough so that $j \geq \sup\{i \in Y : \gamma_i < \xi\}$ and there are $i_0, i_1 < j$ so that $n_{i_0} = n_{i_1}$. Then p_j^* is a condition, and so is

$$p_* := p_j^* \restriction (\kappa \setminus d^*) \cup \{(\alpha, q_i(\alpha)) : \alpha \in d^*\}$$

for any $i \in Y$. Then $p_* \Vdash \check{x}_\xi \subseteq \dot{y} \wedge \dot{f}(i_0) = \dot{f}(i_1)$.

□

In $V[G]$ we have that $2^{\aleph_0} = \kappa > \omega_1$, and therefore CH fails.

Suppose G is generic for $\mathbb{P}$. For any $y \in [\omega_1]^{\omega_1}$ in the final model $V[G]$, there are conditions $p_\alpha \in G$, $\alpha < \omega_1$, so that p_α forces a value for the α element of y. Each p_α only uses a countable support, so is contained in some initial segment of the forcing and therefore y is considered by the bookkeeping at some stage. This, together with an easy density argument, gives that for any $y \in [\omega_1]^{\omega_1}$ in the final model $V[G]$, there is some $\beta < \kappa$ so that $\mathbb{Q}_\beta = \mathrm{Thread}(y)$. For every $\xi < \omega_1$ there is an element in the $\mathbb{Q}_\beta$-generic over $V[G\restriction\beta]$ with supremum $\geq \xi$. Therefore, the union of the $\mathbb{Q}_\beta$-generic is an uncountable subset of y, and each of its initial segments is in V.

Since CH holds in the ground model, there are only $\aleph_1$-many countable subsets of ω_1 in V. Therefore, we have proven

Claim 3.5. *In $V[G]$, $[\omega_1]^{<\omega_1} \cap V$ is a* Superstick *sequence.*

This completes the proof.

Remark 3.6. As in the construction in Section 5 of [3], we have MA(countable) in $V[G]$.

$\square$

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