TIGHT STATIONARITY AND TREE-LIKE SCALES

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Abstract. Let κ be a singular cardinal of countable cofinality, $\langle \kappa_n : n < \omega \rangle$ a sequence of regular cardinals which is increasing and cofinal in κ. Using a scale, we define a mapping μ from $\prod_n \mathcal{P}(\kappa_n)$ to $\mathcal{P}(\kappa^+)$ which relates tight stationarity on κ to the usual notion of stationarity on κ^+. We produce a model where all subsets of κ^+ are in the range of μ for some κ a singular. Using a version of the diagonal supercompact Prikry forcing, we obtain such a model where κ is strong limit. Then we construct a sequence of stationary sets that is not tightly stationary in a strong way, namely, its image under μ is empty. All of these results start from a model with a continuous tree-like scale on κ.

1. Introduction

In their study of the non-saturation of the nonstationary ideal on $[\kappa]^\omega$ for κ a singular cardinal, Foreman and Magidor [8] introduced two concepts of stationarity for singular cardinals (even those of countable cofinality): *mutual stationarity* and *tight stationarity*. Each of these notions is a property of sequences $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ where $S_\xi \subseteq \kappa_\xi$ and $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ is a sequence of regular cardinals cofinal in κ. Tight stationarity is a more tractable strengthening of mutual stationarity that admits analogues of results for the classical notion of stationarity for regular cardinals, namely Fodor's lemma and Solovay's splitting theorem (whether those results hold for mutual stationarity is an open problem, see [7]).

This paper explores a method to transfer results from the theory of stationary subsets of κ^+ to that of tightly stationary sequences on κ. We introduce a function μ which takes a sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ to a subset of κ^+. The key property of μ is that it preserves stationarity in the sense that $\vec{S}$ is tightly stationary if and only if $\mu(\vec{S})$ is stationary (this requires certain assumptions, see Lemma 2.5 for a precise statement). This function μ will be defined from a scale, and makes sense if there is a scale on $\prod \kappa_\xi$ modulo the ideal of bounded subsets of κ.

The existence of μ is by itself enough for some connections between stationarity at κ^+ and tight stationarity at κ. For example, it can be used to derive the version of Fodor's lemma previously obtained by Foreman-Magidor [8] for tight stationarity at κ from the usual Fodor's lemma at κ^+; see Proposition 2.7.

But for other applications, we want to have an inverse for μ, in the following strong sense: for each $A \subseteq \kappa^+$ we want to have a sequence $\vec{S}$ so that $\mu(\vec{S}) = A$ and $\mu(\vec{S}') = \kappa^+ \setminus A$, where $\vec{S}'$ is the sequence $S'_\xi = \kappa_\xi \setminus S_\xi$. Call $A \subseteq \kappa^+$ *careful* if there

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exists such a sequence $\vec{S}$. The notion of carefulness can be thought of as a symmetrical strengthening of being in the range of μ — Boolean operations on careful sets commute with μ, although this is not generally true for sets which are just in the range of μ. Consequently, μ gives a particularly useful connection between careful subsets of κ^+ and sequences of the kind considered for tight stationarity.

If every subset of κ^+ is careful, then we can transfer Solovay's splitting theorem on κ^+ to the context of tight stationarity on κ. Under this assumption, we obtain a new splitting result for tightly stationary sets (Proposition 2.8). We remark that Proposition 2.8 differs from the splitting theorem obtained by Foreman and Magidor in [8].

Although there are many situations in which there exists a non-careful subset of κ^+, the main constructions in this paper show that it is actually consistent for every subset of κ^+ to be careful. In Section 3, we use forcing to construct a model where every subset of κ^+ is careful. The construction succeeds when κ has cofinality which is either countable or indestructibly supercompact. The μ function here is defined from a scale which is *tree-like*, a useful property studied by Pereira [11].

In Section 4, we start with a supercompact cardinal κ and modify the construction of Section 3 so that in the extension, κ is a strong limit singular cardinal of countable cofinality and every subset of κ^+ is careful. Additionally, collapses can be interleaved into the construction so that κ is the least cardinal fixed point (i.e., the least κ with $\kappa = \aleph_\kappa$). This uses ideas from the diagonal supercompact Prikry forcing of Gitik–Sharon [9].

In Section 5, we address the question of whether there is always a sequence of stationary sets that is not tightly stationary. We prove that if the scale used to define μ is tree-like, then there is a sequence $\vec{S}$ such that S_ξ is stationary for every $\xi < \mathrm{cf}(\kappa)$ and $\mu(\vec{S}) = \emptyset$ (in fact, $\mu(\vec{S'}) = \kappa^+$, where $S'_\xi = \kappa_\xi \setminus S_\xi$). This shows in particular that there is a sequence of stationary subsets which is not tightly stationary, under the seemingly mild assumption of a continuous tree-like scale at κ.

2. Preliminaries

First we will define the terminology used in the introduction. Let κ be a singular cardinal, and $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ a sequence of regular cardinals cofinal in κ.Take $\theta = (2^{2^\kappa})^+$ and let $\mathcal{A}$ be an algebra on $H(\theta)$, i.e., a structure on $H(\theta)$ with countably many functions in the language. If $M \prec \mathcal{A}$ is an elementary substructure, then define the *characteristic function of* M as $\chi_M : \xi \mapsto \sup(M \cap \kappa_\xi)$. We say M is *tight* if $M \cap \prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$ is cofinal in $\prod(M \cap \kappa_\xi)$.

Suppose $S_\xi \subseteq \kappa_\xi$ for all $\xi < \mathrm{cf}(\kappa)$. The sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ is *mutually stationary* if for any algebra $\mathcal{A}$ on $H(\theta)$ there is $M \prec \mathcal{A}$ such that $\{\xi : \chi_M(\xi) \notin S_\xi\}$ is bounded in $\mathrm{cf}(\kappa)$ (we say that χ_M *meets* $\vec{S}$). The sequence $\vec{S}$ is *tightly stationary* if for every $\mathcal{A}$ on $H(\theta)$, a tight structure $M \prec \mathcal{A}$ as in the previous definition can be chosen.

For our purposes, a *scale* is a sequence $\langle f_\alpha : \alpha < \kappa^+\rangle$ which is increasing and cofinal in $(\prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi, <^*)$, where $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ are regular cardinals cofinal in κ and $f <^* g$ if and only if $\{\xi : f(\xi) \geq g(\xi)\}$ is bounded in $\mathrm{cf}(\kappa)$. Scales were previously considered in the context of mutual and tight stationarity in [5] and [6]. A basic result of pcf theory due to Shelah [12] says that for singular κ, there is an increasing sequence of regular cardinals $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ which carries a scale. The scales in this paper will always be continuous, which means that for every $\beta < \kappa^+$ of cofinality $> \mathrm{cf}(\kappa)$, if there is an exact upper bound for $\langle f_\alpha : \alpha < \beta\rangle$ (i.e., a $<^*$-upper bound g such that $\langle f_\alpha : \alpha < \beta\rangle$ is cofinal in $\prod_\xi g(\xi)$) then f_β is such a bound.

It is an easy and useful fact that if $\langle f_\alpha : \alpha < \beta\rangle$ is a $<^*$-increasing sequence with $\mathrm{cf}(\beta) > \mathrm{cf}(\kappa)$ and there are $Y \subset \beta$ unbounded and $\xi^* < \mathrm{cf}(\kappa)$ such that $\langle f_\gamma(\xi) : \gamma \in Y\rangle$ strictly increasing for all $\xi^* \leq \xi < \mathrm{cf}(\kappa)$, then the pointwise supremum $\sup_{\gamma<\beta} f_\gamma$ is an exact upper bound. Given a scale $\langle f_\alpha : \alpha < \kappa^+\rangle$ on $\prod_\xi \kappa_\xi$, a *good point* is an ordinal β so that $\langle f_\alpha : \alpha < \beta\rangle$ satisfies the conditions of the previous sentence. A scale is called good if the set of its good points is club in κ^+. In [12] it is shown that the set of good points of a fixed cofinality greater than $\mathrm{cf}(\kappa)$ is stationary.

As stated in the introduction, we can use the scale to relate sequences on the $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ to subsets of κ^+. The key point is Theorem 5.2 of [4], which says that for tight N containing $\langle f_\alpha : \alpha < \kappa^+\rangle$, there is an α such that $\chi_N =^* f_\alpha$. Thus, we can replace characteristic functions of tight structures by scale functions.

Definition 2.1. Suppose $S_\xi \subseteq \kappa_\xi$ for each $\xi < \mathrm{cf}(\kappa)$. Then define

$$\mu(\vec{S}) = \{\alpha : f_\alpha \text{ meets } \vec{S}\}.$$

Let $S'_\xi = \kappa_\xi \setminus S_\xi$. Then define $\nu(\vec{S}) = \kappa^+ \setminus \mu(\langle S'_\xi\rangle)$.

Another way to think of $\nu(\vec{S})$ is $\{\alpha : f_\alpha(\xi) \in S_\xi$ for unboundedly many $\xi\}$.

We list some straightforward algebraic properties of μ.

Proposition 2.2. *Let $S_\xi, T_\xi \subseteq \kappa_\xi$ for $\xi < \mathrm{cf}(\kappa)$. Then*

$$\mu(\langle S_\xi \cap T_\xi\rangle) = \mu(\vec{S}) \cap \mu(\vec{T})$$

and

$$\mu(\langle S_\xi \cup T_\xi\rangle) \supseteq \mu(\vec{S}) \cup \mu(\vec{T}).$$

Recall the notion of a careful subset of κ^+ from the introduction.

Definition 2.3. A set $A \subseteq \kappa^+$ is *careful* if there is a sequence $\vec{S}$ with $\mu(\vec{S}) = \nu(\vec{S}) = A$.

So a careful set A is in the range of the μ function, witnessed by a sequence $\vec{S}$ which does not intersect scale functions indexed by $\kappa^+ \setminus A$ too much. Sequences which witness carefulness behave nicely under finite coordinatewise intersections and unions.

Proposition 2.4. *Let A, B be careful, witnessed by the sequences $\langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ and $\langle T_\xi : \xi < \mathrm{cf}(\kappa)\rangle$, respectively. Then*

$$\mu(\langle S_\xi \cap T_\xi\rangle) = \nu(\langle S_\xi \cap T_\xi\rangle) = A \cap B$$

and

$$\mu(\langle S_\xi \cup T_\xi\rangle) = \nu(\langle S_\xi \cup T_\xi\rangle) = A \cup B.$$

The following lemma is the key point relating tight stationarity to the μ of Definition 2.1. We will work in the case where there is some regular cardinal $\eta < \kappa_0$ so that $S_\xi \subseteq \mathrm{Cof}(\eta)$—this gives the uniformity we need to apply results about the characteristic functions of tight structures from Cummings–Foreman–Magidor [4].

Lemma 2.5. *Let η be an uncountable regular cardinal in the interval $(\mathrm{cf}(\kappa), \kappa_0)$. Suppose $S_\xi \subseteq \kappa_\xi \cap \mathrm{Cof}(\eta)$. Then $\vec{S}$ is tightly stationary iff $\mu(\vec{S}) \cap \mathrm{Cof}(\eta) \cap \mathrm{Good}$ is stationary in κ^+, where Good is the set of good points.*

Proof. If $\vec{S}$ is tightly stationary, then for any algebra $\mathfrak{A}$ on $H(\theta)$ there is a tight $N \prec \mathfrak{A}$ which meets $\vec{S}$ and contains the scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ as an element. Let $\alpha := \sup(N \cap \kappa^+)$. By Theorem 5.2 of [4], $\chi_N =^* f_\alpha$ and α is a good point of cofinality η, so $\alpha \in \mu(\vec{S}) \cap \mathrm{Cof}(\eta) \cap \mathrm{Good}$.

For the converse, suppose that $B = \mu(\vec{S}) \cap \mathrm{Cof}(\eta) \cap \mathrm{Good}$ is stationary in κ^+. Let $C \subseteq [H(\theta)]^{<\eta^+}$ be an arbitrary club so that every member of C is an elementary submodel of $(H(\theta); \in, \langle f_\alpha : \alpha < \kappa^+ \rangle, \vec{S})$. Then construct $\langle M_x : x \in [\kappa^+]^{<\eta} \rangle \subseteq C$ so that:

(1) $x \subseteq M_x$,
(2) if y end-extends x, then $M_x \subseteq M_y$ and M_y contains some $\alpha < \kappa^+$ so that $\chi_{M_x} <^* f_\alpha$.

Define $g : [\kappa^+]^{<\eta} \to \kappa^+$ by sending x to the least α so that $\chi_{M_x} <^* f_\alpha$. Consider the set $D = \{\alpha \in \kappa^+ : g``[\alpha]^{<\eta} \subseteq \alpha\}$, and let $E \subseteq \kappa^+$ be its closure. Then $E \cap \mathrm{Cof}(\eta) = D \cap \mathrm{Cof}(\eta)$, so there exists $\gamma \in B \cap D \cap \mathrm{Cof}(\eta)$. Since γ is good, there are $\langle \gamma_i : i < \eta \rangle$ cofinal in γ and $\xi^* < \mathrm{cf}(\kappa)$ so that for all $\xi \geq \xi^*$, $\langle f_{\gamma_i}(\xi) : i < \eta \rangle$ is strictly increasing. Since $\gamma \in D$, we can further assume that $g(\langle \gamma_j : j < i \rangle) < \gamma_{i+1}$ for all $i < \eta$.

For convenience, let N_i denote the substructure $M_{\langle \gamma_j : j < i \rangle}$. Put $M = \bigcup_{i<\eta} N_i$. Then $M \in C$ (since it is the increasing union of members of C) and M is tight (which follows from clause (2) in the construction of the M_x). The argument is finished by showing that $\chi_M =^* f_\gamma$.

By clause (1) of the construction of the M_x, we have that the range of f_{γ_i} is contained in N_{i+1} for all $i < \eta$, and therefore $f_{\gamma_i} < \chi_{N_{i+1}}$. Since $g(\langle \gamma_j : j < i \rangle) < \gamma_{i+1}$, we also have that $\chi_{N_i} <^* f_{\gamma_{i+1}}$. Putting the inequalities together with the fact that $\eta > \mathrm{cf}(\kappa)$, we have $\chi_M =^* \sup_{i<\eta} \chi_{N_i} =^* \sup_{i<\eta} f_{\gamma_i} =^* f_\gamma$. □

The following proposition is straightforward.

Proposition 2.6. *If S_ξ is club in κ_ξ for all $\xi < \mathrm{cf}(\kappa)$, then $\mu(\vec{S})$ contains a club. If S_ξ is nonstationary in κ_ξ for all ξ, then $\mu(\vec{S})$ is nonstationary.*

Using this point of view offers another proof of the version of Fodor's Lemma for tightly stationary sets on $\aleph_\omega$ proved in [8], which illustrates the definitions here and shows the relationship with the usual Fodor's Lemma on regular cardinals. The proof given below uses the unnecessary assumption that there is a scale of length $\aleph_{\omega+1}$ on $\prod \aleph_n$, but this can be eliminated using the theory of pcf generators.

Proposition 2.7. *Let $k < \omega$. Suppose $\langle S_n : k < n < \omega \rangle$ is tightly stationary and $S_n \subseteq \mathrm{Cof}(\omega_k)$. If $f : \aleph_\omega \to \aleph_\omega$ satisfies $f(\gamma) < \gamma$ for all γ, then there is a function $g \in \prod_{n \in \omega} \aleph_n$ such that the sequence $\langle S_n^g : k < n < \omega \rangle$ defined by $S_n^g = \{\gamma \in S_n : f(\gamma) < g(n)\}$ is tightly stationary.*

Proof. Let $A = \mu(\vec{S}) \cap \{\alpha : \alpha \text{ is a good point of cofinality } \omega_k\}$. This is stationary by Lemma 2.5 since $\vec{S}$ is tightly stationary. Define $\bar{F} : A \to \kappa^+$ to be $\bar{F}(\alpha) =$ least $\beta < \alpha$ such that $f \circ f_\alpha <^* f_\beta$. Such exists since f is regressive and any $\alpha \in A$ is a good point. Then $\bar{F}$ is a regressive function on A, hence by the usual Fodor's lemma, is constant on a stationary set A', say with constant value β_0. Put $g = f_{\beta_0}$. Consider

$$S_n^g = \{\gamma \in S_n : f(\gamma) < g(n)\}.$$

We now show that $A' \subseteq \mu(\langle S_n^g \rangle)$, hence by Lemma 2.5 that $\langle S_n^g \rangle$ is tightly stationary. For any $\alpha \in A'$, we have $f \circ f_\alpha <^* g$ by choice of A' and β_0. This means there is $i \in \omega$ such that for all $n \geq i$ we have $f_\alpha(n) \in S_n^g$, or in other words, $\alpha \in \mu(\langle S_n^g \rangle)$. □

If every subset of κ^+ is careful, then similar ideas can be applied to splitting tightly stationary sets.

Proposition 2.8. *Suppose every subset of κ^+ is careful. Then for any tightly stationary sequence $\vec{S}$, there are $\vec{T}^\xi = \langle T_n^\xi : n < \omega \rangle$ for $\xi < \kappa^+$ such that*

- *$T_n^\xi \subseteq S_n$ for all $n < \omega$, $\xi < \kappa^+$,*
- *$\vec{T}^\xi$ is tightly stationary for all $\xi < \kappa^+$,*
- *$\nu(\langle T_n^\xi \cap T_n^\zeta \rangle) = \emptyset$ for all $\xi \neq \zeta < \kappa^+$.*

Proof. Let $A = \mu(\vec{S}) \cap \mathrm{Good}$, which is stationary in κ^+ by Lemma 2.5. Then A can be split into κ^+ many pairwise disjoint stationary subsets of κ^+, say $\langle A_\xi : \xi < \kappa^+ \rangle$. Each A_ξ is careful, so let $\vec{T}^\xi$ be the corresponding sequence, which is tightly stationary by Lemma 2.5. By intersecting with $\vec{S}$, we may assume that $T_n^\xi \subseteq S_n$ for all $n < \omega$, $\xi < \kappa^+$. By Proposition 2.2, condition (3) holds. □

In the next section, we will show that the hypothesis of Proposition 2.8 is consistent.

From a scale, one can define two-place functions $[\kappa^+]^2 \to \mathrm{cf}(\kappa)$ which will help describe how the μ function works.

Definition 2.9. Let $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a scale on κ, and suppose $\alpha < \beta$. Then $d(\alpha, \beta) = \sup\{\xi + 1 : f_\alpha(\xi) \geq f_\beta(\xi)\}$, and $d^*(\alpha, \beta) = \sup\{\xi + 1 : f_\alpha(\xi) = f_\beta(\xi)\}$.

The function d was used by Shelah in [12], for example, to prove $\kappa^+ \not\to [\kappa^+]^2_{\mathrm{cf}\kappa}$ for singular κ.

A crucial concept for the constructions in this paper is a tree-like scale. This concept appears in Shelah [12] (see II Conclusion 3.5) and was isolated and further studied by Pereira in [11].

Definition 2.10. A scale $\langle f_\alpha : \alpha < \lambda \rangle$ is *tree-like* if whenever $f_\alpha(\xi) = f_\beta(\xi)$, then $f_\alpha \restriction \xi = f_\beta \restriction \xi$.

IF $\mathrm{cf}(\kappa) = \omega$, then any product which carries a scale also carries a tree-like scale, but that scale is not necessarily continuous, as we require. Pereira described a forcing notion in [11] which produces a continuous tree-like scale and preserves cardinals, and hence also the approachability property at κ (a principle which implies that every scale on κ is good) if it holds in the ground model.

Unless otherwise indicated, we assume from now on that $\mathrm{cf}(\kappa) = \omega$ for concreteness, although this assumption will only really be essential in Theorem 1. The next lemma describes how the assumption of a tree-like scale affects the d^* function.

Lemma 2.11. *Let $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a tree-like scale. For any $\alpha, \beta, \gamma \in \kappa^+$, the smaller two among $d^*(\alpha, \beta), d^*(\beta, \gamma), d^*(\alpha, \gamma)$ are equal.*

Proof. Assume without loss of generality that $d^*(\alpha,\beta) \leq d^*(\beta,\gamma) \leq d^*(\alpha,\gamma)$. Fix arbitrary $n < d^*(\beta,\gamma)$. Using the tree-like property and $d^*(\beta,\gamma) \leq d^*(\alpha,\gamma)$, we have $f_\alpha(n) = f_\gamma(n)$. But by definition of d^*, $f_\gamma(n) = f_\beta(n)$. Combining the two equations, we get $f_\alpha(n) = f_\beta(n)$, so $d^*(\alpha,\beta) \geq d^*(\beta,\gamma)$. □

In this situation, there is a qualitative difference between d and d^*.

Proposition 2.12. *Let $\langle f_\alpha : \alpha < \kappa^+\rangle$ be a tree-like scale. Then there are disjoint stationary sets $A, B \subseteq \kappa^+$ such that $d^*(\alpha,\beta)$ is constant on $A \times B$.*

Proof. For each α, let $D(\alpha) = \{n : d^*(\alpha,\beta) = n$ for stationarily many $\beta < \kappa^+\}$. Since $\mathrm{cf}(\kappa^+) > \omega$, each $D(\alpha) \neq \emptyset$.

We will find an α such $|D(\alpha)| > 1$. If this does not exist, then for each α let $n(\alpha)$ be the unique element of $D(\alpha)$. Then $C_\alpha = \{\beta : d^*(\alpha,\beta) = n(\alpha)\}$ must contain a club of κ^+. Let C be the diagonal intersection of the C_α, $\alpha < \kappa^+$. Let n_0 be such that $n(\alpha) = n_0$ for κ^+ many $\alpha \in C$. Then let $E = \{\alpha \in C : n(\alpha) = n_0\}$. If $\alpha < \beta$ are members of E, then $d^*(\alpha,\beta) = n(\alpha) = n_0$. This implies that $f_\alpha(n_0), \alpha \in E$, are pairwise distinct, a contradiction.

So fix α_0 such that $|D(\alpha_0)| > 1$, and let $m < n$ be elements of $D(\alpha_0)$. Then let $A = \{\alpha : d^*(\alpha_0,\alpha) = m\}$ and $B = \{\beta : d^*(\alpha_0,\beta) = n\}$. By Lemma 2.11, $d^*(\alpha,\beta) = m$ for all $\alpha \in A, \beta \in B$. □

On the other hand, Shelah [12] showed that if $A, B \subseteq \kappa^+$ are unbounded, then for any sufficiently large n, there are $\alpha \in A$ and $\beta \in B$ such that $d(\alpha,\beta) = n$.

The next lemma gives a combinatorial criterion for carefulness which involves the d^* function.

Lemma 2.13. *Suppose $A \subseteq \kappa^+$ and there is $F : \kappa^+ \to \omega$ such that $d^*(\alpha,\beta) \leq \max\{F(\alpha), F(\beta)\}$ for all $\alpha \in A$ and $\beta \notin A$. Then A is careful.*

Proof. Define $S_n = \{f_\alpha(n) : \alpha \in A$ and $F(\alpha) \leq n\}$. Then $A \subseteq \mu(\vec{S})$ since for any $\alpha \in A$, $f_\alpha(n) \in S_n$ for all $n \geq F(\alpha)$. It remains to show that $\nu(\vec{S}) \subseteq A$. For $\beta \in \kappa^+ \setminus A$, we will show that $f_\beta(n) \notin S_n$ for $n \geq F(\beta)$. Let n be so that $f_\beta(n) \in S_n$. Then $d^*(\alpha,\beta) > n$ for some $\alpha \in A$ with $F(\alpha) \leq n$. Since $n < d^*(\alpha,\beta) < \max\{F(\alpha), F(\beta)\}$, it follows that $n < F(\beta)$. □

Remark 2.14. This is actually an equivalence if the background scale is tree-like: if A is careful, witnessed by $\vec{S}$, then define $F(\alpha)$ to be the least n such that $f_\alpha(n) \in S_n$ if $\alpha \in A$, and the least n such that $f_\alpha(n) \notin S_n$ if $\alpha \in A$.

We conclude this section by identifying situations where there exist subsets of κ^+ which are not in the range of μ. Suppose $2^\kappa < 2^{\kappa^+}$ (e.g., when the SCH holds at κ). Then there are only 2^κ choices for a sequence $\bar{S}$, so there is a subset of κ^+ which is not in the range of μ.

We can also add a set which is not in the range of μ by forcing. This example was inspired by similar arguments of Foreman and Steprāns from Section 4 of [5].

Think of $P = \mathrm{Add}(\omega,\kappa^+)$ as the forcing adding a subset of κ^+ using finite conditions—if G is generic for P, then $\bigcup G$ is a function $\omega \times \kappa^+ \to 2$, and using a bijection φ between κ^+ and $\omega \times \kappa^+$, we obtain a subset S from G (whose characteristic function is $\bigcup G \circ \varphi$). Recall that P is c.c.c., and for any $\lambda < \kappa^+$, $P \simeq \mathrm{Add}(\omega,\lambda) \times \mathrm{Add}(\omega,\kappa^+)$. Since P is c.c.c., any function in $\prod_{n<\omega} \kappa_n \cap V[G]$ is dominated pointwise by a function in $\prod_{n<\omega} \kappa_n \cap V$. Thus the scale $\vec{f}$ in V remains a scale in $V[G]$.

Proposition 2.15. *Let $P = \mathrm{Add}(\omega, \kappa^+)$, and $S \subseteq \kappa^+$ as above. Then in $V[G]$, S is not in the range of μ.*

Proof. Work in $V[G]$. For any sequence $\langle U_n : n < \omega\rangle$ with $U_n \subseteq \kappa_n$, we claim that the sequence (and hence also every U_n) is contained in the generic extension of V by $\mathrm{Add}(\omega, \lambda)$ for some $\lambda < \kappa^+$. This is because there is a nice name for each U_n (i.e., consisting of pairs $(\check{\alpha}, p)$ where for any given $\check{\alpha}$, $\{p : (\check{\alpha}, p) \in \dot{U}_n\}$ is an antichain), so there is a name for $\langle U_n : n < \omega\rangle$ which uses at most κ many elements of P.

Factor $V[G] = V[H][G']$ where H is generic for $\mathrm{Add}(\omega, \lambda)$ and $\langle U_n : n < \omega\rangle \in V[H]$, and G' is generic for the quotient $\mathrm{Add}(\omega, \kappa^+)$. Now $\mu(\langle U_n\rangle)$ lies in $V[H]$. By a density argument using the construction of S from G', $S \notin V[H]$. □

In the next section, we show that it is consistent that every subset of κ^+ is in the range of μ (in fact, careful).

3. A model where every set is careful

A *better* scale is a scale $\langle f_\alpha : \alpha < \kappa^+\rangle$ such that for every limit ordinal $\alpha < \kappa^+$ there is a club $C \subseteq \alpha$ such that for every $\gamma \in C$ there is $N < \omega$ such that $\forall n > N\, (f_\beta(n) < f_\gamma(n))$ for all $\beta < \gamma$ with $\beta \in C$. This is a stronger property than the one that good scales satisfy. The existence of better scales is a consequence of the weak square $\square^*_\kappa$.

We start with the observation that if the background scale is better, then every bounded subset of κ^+ is careful. The argument follows along the lines of the construction of an ADS-sequence from a better scale by Cummings, Foreman and Magidor in [3].

Proposition 3.1. *If $\langle f_\alpha : \alpha < \kappa^+\rangle$ is a better scale, then every bounded $A \subset \kappa^+$ is careful.*

Proof. In [3], it is proved from a better scale that that for every $\gamma < \kappa^+$, there is a function $G_\gamma : \gamma \to \omega$ such that for any $\alpha < \beta < \gamma$, $d^*(\alpha, \beta) < \max\{G_\gamma(\alpha), G_\gamma(\beta)\}$. Now if $A \subset \kappa^+$ is bounded, then let γ be a bound. Set $F(\alpha)$ to be $\max(d(\alpha, \gamma) + 1, G(\alpha))$ if $\alpha < \gamma$, 0 if $\alpha = \gamma$, and $d(\gamma, \alpha) + 1$ if $\alpha > \gamma$. We show that $d^*(\alpha, \beta) \leq \max\{F(\alpha), F(\beta)\}$ for all $\alpha \in A$, $\beta \notin A$. If $\beta < \gamma$, then

$$d^*(\alpha, \beta) < \max\{G(\alpha), G(\beta)\} \leq \max\{F(\alpha), F(\beta)\}.$$

If $\beta = \gamma$, then

$$d^*(\alpha, \beta) < d(\alpha, \beta) \leq F(\alpha).$$

If $\beta > \gamma$, assume towards a contradiction that $d^*(\alpha, \beta) > \max\{F(\alpha), F(\beta)\}$. In particular, this assumption implies that $d^*(\alpha, \beta) > d(\alpha, \gamma), d(\gamma, \beta)$, so $f_\alpha(d^*(\alpha, \beta) - 1) < f_\gamma(d^*(\alpha, \beta) - 1) < f_\beta(d^*(\alpha, \beta) - 1)$, contradicting the definition of d^*. □

Starting from a continuous tree-like scale, we will force so that every subset of κ^+ is careful. We will see below that this poset is c.c.c., and therefore $\langle f_\alpha : \alpha < \kappa^+\rangle$ remains a scale in $V[G]$.

Theorem 1. *Let $\langle f_\alpha : \alpha < \kappa^+\rangle$ be a continuous tree-like scale and $A \subseteq \kappa^+$. Then there is a c.c.c. forcing extension in which A is careful.*

Remark 3.2. In fact, the proof will show that the poset is ω_1-Knaster.

Proof. Given A, define $\mathbb{Q}_A$ to be the forcing of finite functions $p : \kappa^+ \to \omega$ such that $d^*(\alpha, \beta) \leq \max\{p(\alpha), p(\beta)\}$ for any $\alpha \in \mathrm{dom}(p) \cap A$ and $\beta \in \mathrm{dom}(p) \setminus A$, ordered by extension.

Now we will show that $\mathbb{Q}_A$ is c.c.c. Towards a contradiction, suppose $\{p_\xi : \xi < \omega_1\}$ is an uncountable antichain. Using the Δ-system lemma, we may assume that the domains of the p_ξ form a Δ-system. The strategy of the proof is to repeatedly thin the antichain by choosing an uncountable subset with certain nice properties, and without loss of generality renaming the thinned antichain by $\{p_\xi : \xi < \omega_1\}$. At the end we will have thinned enough to see that certain members of the antichain were actually compatible.

For any condition $p \in \mathbb{Q}_A$, let the *type of* p be the ordered pair (m, n), where $m = |\mathrm{dom}(p) \cap A|$ and $n = |\mathrm{dom}(p) \cap (\kappa^+ \setminus A)|$. Thin to assume that all members have the same type (m, n), and that the p_ξ agree on the root of the Δ-system. By throwing away the root from the domain of each condition, we may assume the p_ξ have disjoint domains.

Enumerate $\mathrm{dom}(p_\xi) \cap A$ as $\{\alpha^i_\xi : i < m\}$ and $\mathrm{dom}(p_\xi) \setminus A$ as $\{\beta^i_\xi : i < n\}$. By thinning further we may assume that for every $i < m$, $j < n$, there is $k_{ij} < \omega$ (not depending on ξ) such that $\forall \xi < \omega_1 (d^*(\alpha^i_\xi, \beta^j_\xi) = k_{ij})$. By thinning yet further we can assume that for every $i < m$, $j < n$, either

$$\forall \xi < \omega_1 (p_\xi(\alpha^i_\xi) \geq k_{ij})$$

or

$$\forall \xi < \omega_1 (p_\xi(\beta^j_\xi) \geq k_{ij})$$

(i.e., whether it is α^i or β^j that satisfies this does not depend on ξ).

The goal is to thin the antichain further so that we can find some $i_0 < m$ (or $j_0 < n$) such that $p_\xi(\alpha^{i_0}_\xi) \geq d^*(\alpha^{i_0}_\xi, \beta^j_\zeta)$ for all $\xi, \zeta < \omega_1, j < n$ (or $p_\xi(\beta^{j_0}_\xi) \geq d^*(\alpha^i_\zeta, \beta^{j_0}_\xi)$ for all $\zeta < \omega_1, i < m$). Thus the incompatibility between different members of the antichain cannot come from the elements $\alpha^{i_0}_\xi$ (or $\beta^{j_0}_\xi$) of the domain of each condition, so the property of being an antichain is preserved if we remove these elements from the domain of each condition. Repeating this process, we eventually reach an uncountable antichain where every member is of the same type $(m, 0)$ or $(0, n)$, a contradiction since these would all be compatible in $\mathbb{Q}_A$.

Choose $i_0 < m$ and $j_0 < n$ so that $k_{i_0 j_0} = \max_{i<m, j<n} k_{ij}$, and let $M = k_{i_0 j_0}$. We handle the case $\forall \xi < \omega_1 (p_\xi(\alpha^{i_0}_\xi) \geq M)$, the case with β is similar. To avoid a mess of sub- and superscripts, we denote $\alpha^{i_0}_\xi$ by α_ξ.

We will perform the thinning one j at a time, so fix $j < n$. It suffices to show that there is an uncountable set $Z \subset \omega_1$ such that for all $\xi, \zeta \in Z$, $p_\xi(\alpha_\xi) \geq d^*(\alpha_\xi, \beta^j_\zeta)$.

Claim 3.3. *For every $\xi, \zeta < \omega_1$, either*

$$d^*(\alpha_\xi, \beta^j_\zeta) \leq k_{i_0 j} \text{ and } d^*(\alpha_\zeta, \beta^j_\xi) \leq k_{i_0 j} \tag{1}$$

or

$$d^*(\alpha_\xi, \alpha_\zeta) = k_{i_0 j} \text{ and the first case fails.} \tag{2}$$

Proof. Suppose the first case fails. Without loss of generality, $d^*(\alpha_\xi, \beta^j_\zeta) > k_{i_0 j}$. Since $d^*(\alpha_\zeta, \beta^j_\zeta) = k_{i_0 j}$, Lemma 2.11 implies that $d^*(\alpha_\xi, \alpha_\zeta) = k_{i_0 j}$. □

Color $[\omega_1]^2$ in two colors, where $\{\xi, \zeta\}$ is colored according to which case of Claim 3.3 holds. Now apply the Dushnik–Miller theorem, $\omega_1 \to (\omega_1, \omega)^2$. In the first possibility, there is an uncountable set X such that (1) holds between every $\xi, \zeta \in X$. Then we are done, since by choice of i_0, for all $\xi \in X$,

$$p_\xi(\alpha_\xi^{i_0}) \geq M \geq k_{i_0 j}.$$

In the second possibility, there is an infinite set Y such that (2) holds between every $\xi, \zeta \in Y$. In particular, (1) fails for every $\xi, \zeta \in Y$, so by Ramsey's theorem there is an infinite Y' such that either

$$d^*(\alpha_\xi, \beta_\zeta^j) > k_{i_0 j} \text{ for all } \xi < \zeta \text{ in } Y',$$

or

$$d^*(\alpha_\zeta, \beta_\xi^j) > k_{i_0 j} \text{ for all } \xi < \zeta \text{ in } Y'.$$

Assume that $d^*(\alpha_\xi, \beta_\zeta^j) > k_{i_0 j}$ for $\xi < \zeta$ in Y'; the other possibility of Ramsey's theorem would proceed similarly.

Fix $\xi < \zeta < \nu \in Y'$. Then $d^*(\alpha_\xi, \beta_\nu^j) > k_{i_0 j}$ and $d^*(\alpha_\zeta, \beta_\nu^j) > k_{i_0 j}$. By Lemma 2.11, we have $d(\alpha_\xi, \alpha_\zeta) > k_{i_0 j}$, but this contradicts (2). Theorem 1 is proved. □

Corollary 3.4. *There is a c.c.c. forcing extension in which every subset of κ^+ is careful.*

Proof. Iterate the forcing from Theorem 1 using finite support, with the usual bookkeeping to take care of any sets that were added in the construction. □

The proof of Theorem 1 relied heavily on the fact that $\mathrm{cf}(\kappa) = \omega$ (and that $\mathbb{P}$ used finite conditions). We can generalize Theorem 1 to singular cardinals with measurable cofinality, and Corollary 3.4 to singular cardinals with supercompact cofinality.

Theorem 2. *Let κ be a singular cardinal with $\mathrm{cf}(\kappa) = \theta$ and $\theta < \kappa$ be an indestructibly supercompact cardinal. Let $\langle \kappa_i : i < \theta \rangle$ be a sequence of regular cardinals cofinal in κ and $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a continuous tree-like scale on $\prod_i \kappa_i$. Then there is poset which is $< \theta$-directed closed and θ^+-c.c. forcing that every subset of κ^+ is careful.*

Proof. Given $A \subseteq \kappa^+$, define $\mathbb{Q}_A$ to be the forcing of partial functions $p : \kappa^+ \to \theta$ with $|\mathrm{dom}(p)| < \theta$ such that $d^*(\alpha, \beta) \leq \max\{p(\alpha), p(\beta)\}$ for any $\alpha \in \mathrm{dom}(p) \cap A$ and $\beta \in \mathrm{dom}(p) \setminus A$, ordered by extension. The poset $\mathbb{Q}_A$ is clearly $< \theta$-directed closed.

Iterate the posets $\mathbb{Q}_A$ with supports of size $< \theta$, using a suitable bookkeeping to ensure that for each A in the final model, $\mathbb{Q}_A$ was used at some stage. The indestructibility of the supercompactness of θ is used in order to ensure that θ is supercompact in all of the models along the iteration. Let $\mathbb{P}$ denote the iteration poset, and $\dot{Q}_\gamma$ name $\mathbb{Q}_{A^\gamma}$, where $A^\gamma \in V^{\mathbb{P}\restriction\gamma}$ is the set being made careful at stage γ. It is clear that $\mathbb{P}$ is $< \theta$-directed closed, so it remains to check that $\mathbb{P}$ is θ^+-c.c. Since it is not true in general that an iteration of θ^+-c.c. posets using $< \theta$ supports is θ^+-c.c., we will argue for the whole iteration poset instead of the individual factors.

For contradiction, fix an antichain $\{p_\xi : \xi < \theta^+\}$. By θ-distributivity, there is a dense set of conditions p in the iteration where for each $\gamma \in \mathrm{dom}(p)$, $p\restriction\gamma$ forces the values of $p(\gamma)$ and $\{\alpha \in \mathrm{dom}(p(\gamma)) : \alpha \in A^\gamma\}$ (these are in the ground model). We will assume that the elements of the antichain were taken from this dense set. For $\xi < \theta^+$, let the *type of* p_ξ *at* γ be the ordered pair (m, n), where $m = |\mathrm{dom}(p(\gamma)) \cap A^\gamma|$ and $n = |\mathrm{dom}(p(\gamma)) \setminus A^\gamma|$ (by restricting to the dense set, this can be computed in V). By judicious thinning, we may assume that the supports of the p_ξ form a Δ-system with root S, and for each $\gamma \in S$,

- all of the $p_\xi(\gamma)$ have the same type, so we can enumerate $\mathrm{dom}(p_\xi(\gamma)) \cap A^\gamma$ as $\{\alpha^i_\xi : i < m\}$ and $\mathrm{dom}(p_\xi) \setminus A^\gamma$ as $\{\beta^i_\xi : i < n\}$,
- the domains of the $p_\xi(\gamma)$ form a Δ-system, the $p_\xi(\gamma)$ agree on the root, and the $p_\xi \restriction \gamma$ force the same members of the root into A^γ.
- for every $i < m$, $j < n$, there is $k_{ij} < \theta$ (not depending on ξ) such that $\forall \xi < \omega_1 (d^*(\alpha^i_\xi, \beta^j_\xi) = k_{ij})$,
- for every $i < m$, $j < n$, either
$$\forall \xi < \theta^+ (p_\xi(\gamma)(\alpha^i_\xi) \geq k_{ij})$$
or
$$\forall \xi < \theta^+ (p_\xi(\gamma)(\beta^j_\xi) \geq k_{ij})$$
(i.e., whether it is α^i or β^j that satisfies this does not depend on ξ).

These assumptions are analogous to ones we made in the proof of Theorem 1.

For distinct $\xi, \zeta < \theta^+$, let $\gamma(\xi, \zeta)$ be the least γ such that $p_\xi(\gamma)$ and $p_\zeta(\gamma)$ are incompatible. Note that $\gamma(\xi, \zeta) \in S$ for every ξ, ζ. By Rowbottom's theorem $\theta \to (\theta)^2_{<\theta}$, there is a subset $C \subseteq \theta^+$ of size θ and some γ such that $\gamma(\xi, \zeta) = \gamma$ for all $\xi, \zeta \in C$. By relabeling the elements of the antichain, we may assume that $C = \theta$. Fix $i < m$ and $j < n$. A version of Claim 3.3 holds in this case.

Claim 3.5. *For every $\xi, \zeta < \theta$, either*
$$d^*(\alpha^i_\xi, \beta^j_\zeta) \leq k_{ij} \text{ and } d^*(\alpha^i_\zeta, \beta^j_\xi) \leq k_{ij} \tag{3}$$
or
$$d^*(\alpha^i_\xi, \alpha^i_\zeta) = k_{ij} \text{ and the first case fails.} \tag{4}$$

Color $[\theta]^2$ in two colors, where $\{\xi, \zeta\}$ is colored according to which case of Claim 3.5 holds. Let U be a θ-complete normal ultrafilter on θ. By Rowbottom's theorem, there is $A_{i,j} \in U$ such that either (3) holds for all $\xi, \zeta \in A_{i,j}$, or (4) holds for all $\xi, \zeta \in A_{i,j}$.

By the same reasoning as in Theorem 1, the second possibility cannot occur. Let $A = \bigcap_{i,j} A_{i,j}$. For any distinct $\xi, \zeta \in A$,
$$d^*(\alpha^i_\xi, \beta^j_\zeta) \leq k_{ij} \text{ and } d^*(\alpha^i_\zeta, \beta^j_\xi) \leq k_{ij}$$
for all $i < m$, $j < n$. By our thinning assumptions, for any $i < m, j < n$, either
$$p_\xi(\gamma)(\alpha^i_\xi), p_\zeta(\gamma)(\alpha^i_\zeta) \geq k_{ij}$$
or
$$p_\xi(\gamma)(\beta^j_\xi), p_\zeta(\gamma)(\beta^j_\zeta) \geq k_{ij}.$$
In either case, it follows that $d^*(\alpha^i_\xi, \beta^j_\zeta) \leq \max\{p_\xi(\gamma)(\beta^j_\xi), p_\zeta(\gamma)(\beta^j_\zeta)\}$ and that $d^*(\alpha^i_\zeta, \beta^j_\xi) \leq \max\{p_\xi(\gamma)(\beta^j_\xi), p_\zeta(\gamma)(\alpha^i_\zeta)\}$. By the minimality of γ, $p_\xi \restriction \gamma$ and $p_\zeta \restriction \gamma$ are compatible and any common extension forces that $p_\xi(\gamma)$ and $p_\zeta(\gamma)$ are compatible. $\square$

Remark 3.6. To prove that the individual posets $\mathbb{P}_A$ as above are θ^+-c.c., it is enough for θ to be measurable.

4. All sets careful and κ strong limit

In the model produced by the forcing of Theorem 1, $2^\omega > \kappa$. However, in singular cardinal combinatorics, the case where the singular cardinal κ is strong limit is of particular interest. Large cardinals are required to obtain a model where every set is careful and κ is strong limit, as the SCH would fail at κ in such a model. Using a supercompact cardinal, we have the following:

Theorem 3. *Let κ be an indestructibly supercompact cardinal and $\mu = \kappa^{+\kappa+1}$. Then there is a forcing poset which preserves cardinals below κ and above μ, and adds no bounded subsets of κ, such that in the extension:*

- *κ is a singular strong limit cardinal with countable cofinality, and $\mu = \kappa^+$,*
- *there is a continuous scale on κ of length μ for which every subset of μ is careful.*

For simplicity of our arguments, assume GCH holds above κ in the ground model. By some preliminary forcing using slight modifications of Theorem 1 of Cummings [1], we arrange so that there is a continuous tree-like scale $\langle G_\alpha : \alpha < \mu\rangle$ on $\prod_{\xi<\kappa} \kappa^{+\xi+1}$ (modulo the bounded ideal on κ). Using Theorem 17 of [3], we can also arrange that $\langle G_\alpha : \alpha < \mu\rangle$ is a good scale.

Our plan is to make every subset of μ careful relative to $\langle G_\alpha : \alpha < \mu\rangle$, and then use a diagonal Prikry forcing technique from Gitik–Sharon [9] to singularize κ while reflecting the scale down to κ (as in Cummings–Foreman [2]). Let X_ξ be the set of $x \in [\kappa^{+\xi+1}]^{<\kappa}$ with $\kappa_x := x \cap \kappa$ an inaccessible cardinal less than κ and $\mathrm{ot}(x \cap \kappa^{+\zeta+1}) = \kappa_x^{+\zeta+1}$ for all $\zeta \leq \xi$. Then define LP to be the set of all finite sequences $\langle x_0, \dots, x_{n-1}\rangle$ satisfying:

- $x_0 \in X_0$.
- For each $i < n$, $x_{i+1} \in X_{\kappa_{x_i}}$.
- $x_i \subseteq x_{i+1}$ and $\mathrm{ot}(x_i) < \kappa_{x_{i+1}}$ (we abbreviate this condition as $x_i \precneqq x_{i+1}$).

This will be the set of all "lower parts" of conditions in a future Prikry forcing. The posets we define below will be κ-distributive and therefore all models will compute LP in the same way.

4.1. Carefulizing forcing.

To make every subset of μ careful, we will define a poset $\mathbb{P}$ akin to those of Theorems 1 and 2. One challenge is that in addition to making ground model subsets of μ careful, we must also anticipate subsets added by the Prikry forcing.

For each family $\vec{A} = \langle A_s : s \in \mathrm{LP}\rangle$, $A_s \subseteq \mu$, define $\mathbb{Q}_{\vec{A}}$ to be the poset of partial functions $P : \mathrm{LP} \times \mu \to \kappa$ such that:

1. $|\mathrm{dom}(P)| < \kappa$, and if t extends s and $(t, \alpha) \in \mathrm{dom}(P)$, then also $(s, \alpha) \in \mathrm{dom}(P)$.
2. If (s, α) and (s, β) are in $\mathrm{dom}(P)$ with $\alpha \in A_s$ and $\beta \notin A_s$, then $d^*_G(\alpha, \beta) \leq \max\{P(s, \alpha), P(s, \beta)\}$. (Here d^*_G is just the d^* function on the scale $\vec{G}$.)
3. If $(t, \alpha) \in \mathrm{dom}(P)$, $s \subseteq t$, and $\alpha \in A_u$ for all $s \subseteq u \subseteq t$, then $P(t, \alpha) = P(s, \alpha)$.
4. If $(t, \alpha) \in \mathrm{dom}(P)$, $s \subseteq t$, and $\alpha \notin A_u$ for all $s \subseteq u \subseteq t$, then $P(t, \alpha) = P(s, \alpha)$.

The ordering on $\mathbb{Q}_{\vec{A}}$ is function extension.

By the usual bookkeeping argument, we can define $\mathbb{P}$, an iteration of posets $\mathbb{Q}_{\vec{A}}$ using supports of size $< \kappa$ so that in the generic extension by $\mathbb{P}$, for each family $\vec{A} = \langle A_s : s \in \mathrm{LP}\rangle$ of subsets of μ indexed by LP, there is a function $F : \mathrm{LP} \times \mu \to \kappa$ such that:

- If (s, α) and (s, β) are in $\mathrm{dom}(F)$ with $\alpha \in A_s$ and $\beta \notin A_s$, then $d^*_G(\alpha, \beta) \leq \max\{F(s, \alpha), F(s, \beta)\}$.
- If $(t, \alpha) \in \mathrm{dom}(F)$, $s \subseteq t$, and $\alpha \in A_u$ for all $s \subseteq u \subseteq t$, then $F(t, \alpha) = F(s, \alpha)$.
- If $(t, \alpha) \in \mathrm{dom}(F)$, $s \subseteq t$, and $\alpha \notin A_u$ for all $s \subseteq u \subseteq t$, then $F(t, \alpha) = F(s, \alpha)$.

We now check that $\mathbb{P}$ does not collapse cardinals. It is easy to see that $\mathbb{P}$ is $< \kappa$-directed closed.

Lemma 4.1. *$\mathbb{P}$ is κ^+-c.c.*

Proof of Lemma 4.1. Suppose $P_\xi : \xi < \kappa^+$ is an antichain. As in the proof of Theorem 2, we will assume that the supports of the conditions form a Δ-system with root S, and that for each γ, $P{\restriction}\gamma$ decides the values of $P(\gamma)$ and $\{(\alpha, t) \in \mathrm{dom}(P(\gamma)) : \alpha \in A^\gamma_t\}$, where $\langle A^\gamma_t : t \in \mathrm{LP}\rangle$ is the family used at stage γ. We will assume that the elements of the antichain were taken from this dense set. We may also assume that for each $\gamma \in S$ the domains of the $P_\xi(\gamma)$ form a Δ-system, and furthermore that the sets $D^\gamma_\xi = \{s \in \mathrm{LP} : \exists\alpha\ (s, \alpha) \in \mathrm{dom}(P_\xi)(\gamma)\}$ form a Δ-system. Let R^γ denote the root of the D^γ_ξ system. For any condition $P \in \mathbb{P}$ and $s \in \mathrm{LP}$, define the s-type of P at γ to be the ordered pair (m, n), where $m = |\{\alpha \in A_s : (s, \alpha) \in \mathrm{dom}(P(\gamma))\}|$ and $n = |\{\alpha \notin A_s : (s, \alpha) \in \mathrm{dom}(P(\gamma))\}|$. By thinning the antichain, we may assume that for each $\gamma \in S$:

- If $s \in R^\gamma$, then all of the P_ξ have the same s-type (m_s, n_s) at γ, so we can enumerate $\{\alpha \in A_s : (s, \alpha) \in \mathrm{dom}(P(\gamma))\}$ as $\{\alpha^{s,i}_\xi : i < m_s\}$ and $\{\alpha \notin A_s : (s, \alpha) \in \mathrm{dom}(P)\}$ as $\{\beta^{s,i}_\xi : i < n_s\}$,
- the $P_\xi(\gamma)$ agree on the common parts of their domains,
- for every $s \in R^\gamma$, and every $i < m_s$, $j < n_s$, there is $k^s_{ij} < \kappa$ (not depending on ξ) such that $\forall \xi < \kappa^+ (d^*(\alpha^{s,i}_\xi, \beta^{s,j}_\xi) = k_{ij})$,
- for every $s \in R^\gamma$, $i < m_s$, $j < n_s$, either
$$\forall \xi < \kappa^+ (P_\xi(\gamma)(s, \alpha^{s,i}_\xi) \geq k_{ij})$$
or
$$\forall \xi < \kappa^+ (P_\xi(\gamma)(s, \beta^{s,j}_\xi) \geq k_{ij})$$
(i.e., whether it is $\alpha^{s,i}$ or $\beta^{s,j}$ that satisfies this does not depend on ξ).

For distinct $\xi, \zeta < \kappa^+$, let $\gamma(\xi, \zeta)$ be the least γ such that $P_\xi(\gamma)$ and $P_\xi(\gamma)$ are incompatible. Note that $\gamma(\xi, \zeta) \in S$ for every ξ, ζ. By Rowbottom's theorem, there is a subset $C \subseteq \kappa^+$ of size κ and some γ such that $\gamma(\xi, \zeta) = \gamma$ for all $\xi, \zeta \in C$.

For $\xi \neq \zeta < \kappa^+$ and every $\gamma \in S$, $P_\xi(\gamma) \cup P_\zeta(\gamma)$ can only fail to be a valid condition in the poset $\mathbb{Q}_{\vec{A}^\gamma}$ by (2) of the definition of the poset: one can check that conditions (3) and (4) are satisfied by using conditions (3) and (4) for $P_\xi(\gamma)$ and $P_\zeta(\gamma)$, together with condition (1) and the fact that the elements of the antichain agree on the common parts of their domains. Therefore we have proven

Claim 4.2. *For $\xi \neq \zeta < \kappa^+$, there is an $s \in R$ and $\alpha, \beta < \mu$ such that exactly one of α, β is in A_s, $(s, \alpha) \in \operatorname{dom}(P_\xi(\gamma))$, $(s, \beta) \in \operatorname{dom}(P_\zeta(\gamma))$, and $d^*_G(\alpha, \beta) > \max\{P_\xi(\gamma)(s, \alpha), P_\zeta(\gamma)(s, \beta)\}$.*

Using Rowbottom's theorem, we have a subset of C of size κ for which there is a single s that sees the incompatibility between its elements. The proof of the lemma is completed exactly as in Theorem 2. □

4.2. **Diagonal Prikry forcing.** Let G be $\mathbb{P}$-generic, and work in $V[G]$. We now define a version of the supercompact diagonal Prikry forcing $\mathbb{R}$. In $V[G]$, κ remains supercompact, so let U be a $\kappa^{+\kappa+1}$-supercompactness measure, i.e., a normal, fine, κ-complete measure on $[\mu]^{<\kappa}$. For $\xi < \kappa$, define a $\kappa^{+\xi+1}$-supercompactness measure U_ξ by

$$X \in U_\xi \quad \text{iff} \quad \{x \in [\mu]^{<\kappa} : x \cap \kappa^{+\xi+1} \in X\} \in U.$$

The measure U_ξ concentrates on the set X_ξ.

Conditions in $\mathbb{R}$ are sequences of the form

$$p = \langle x^p_0, \ldots, x^p_{n-1}\rangle^\frown \langle Y^p_\xi : \kappa_{x^p_{n-1}} \leq \xi < \kappa\rangle$$

for some $n < \omega$ (the *length* of p), where $\langle x_0, \ldots, x_{n-1}\rangle \in \mathrm{LP}$, $\xi_p = 0$ if $n = 0$ and $\xi_p = \kappa_{x^p_{n-1}}$ if $n > 0$, and $Y_\xi \in U_\xi$ for each $\xi_p \leq \xi < \kappa$. When p is clear from the context, we will omit the superscript p and use the abbreviation κ_i for κ_{x_i}. We will call $\langle x_0, \ldots, x_{n-1}\rangle$ the *lower part*, and $\langle Y_\xi : \kappa_{n-1} \leq \xi < \kappa\rangle$ the *upper part* of p.

A condition $q = \langle x^q_0, \ldots, x^q_{m-1}\rangle^\frown \langle Y^q_\xi : \xi_q \leq \xi < \kappa\rangle$ extends p (written $q \leq p$) if and only if

- $m \geq n$, and $x^q_i = x^p_i$ for all $i < n$.
- For each $n \leq i < m$, $x^q_i \in Y^p_{\xi_i}$, where $\xi_i = \kappa_{x^q_{i-1}}$.
- $Y^q_\xi \subseteq Y^p_\xi$ for each $\xi \geq \xi_q$.

As usual in Prikry-type forcings, q *directly extends* p (written $q \leq^* p$) in case $q \leq p$ and q has the same length as p. The underlying set of $\mathbb{R}$ equipped with the $\leq^*$ ordering is $< \kappa$-closed, by the completeness of the ultrafilters.

Lemma 4.3 (Diagonal intersection)**.** *Let $\langle \vec{Y}^s : s \in \mathrm{LP}\rangle$ be a family of upper parts so that $s^\frown \vec{Y}^s \in \mathbb{R}$. Then there is a sequence $\langle Z_\xi : \xi < \kappa\rangle$ such that for every $s \in \mathrm{LP}$, every extension of $s^\frown \langle Z_\xi : \xi_s \leq \xi < \kappa\rangle$ is compatible with $s^\frown \vec{Y}_s$.*

Proof. For each $s \in \mathrm{LP}$, write $\vec{Y}^s = \{Y^s_\xi : \xi_s \leq \xi < \kappa\}$. For each $x \in [\kappa^{+\kappa}]^{<\kappa}$ and $\xi \geq \kappa_x$, define $W^x_\xi := \bigcap\{Y^s_\xi : \bigcup s \subseteq x\}$. Since there are fewer than κ many $s \in \mathrm{LP}$ with $\bigcup s \subseteq x$ for a given $x \in X_\xi$, all of the W^x_ξ are in U_ξ. Now for each $\xi < \kappa$ let $Z_\xi = \{y \in [\kappa^{+\xi+1}]^{<\kappa} : \forall x \in X_\xi\ (x \subsetneq y \to y \in W^x_\xi)\}$, the diagonal intersection of the W^x_ξ, $x \in X_\xi$. By normality of U_ξ, $Z_\xi \in U_\xi$.

We now check that this works. Suppose $t = \{x_0, \ldots, x_{m-1}\}$ is the lower part of an extension of $s^\frown \langle Z_\xi : \xi < \kappa\rangle$ for some $s \in \mathrm{LP}$. For any $i < m$ greater than the length of s, $\bigcup s \subsetneq x_i$, so $x_i \in Y^s_{\xi_i}$. □

In the situation of the lemma, we will call $\langle Z_\xi : \xi < \kappa\rangle$ the *diagonal intersection* of $\langle \vec{Y}^s : s \in \mathrm{LP}\rangle$.

Let $H = \langle x_0, x_1, \ldots\rangle$ be the generic sequence added by $\mathbb{R}$. Note that $x_n \in X_{\xi_n}$, where $\xi_0 = 0$ and $\xi_n = \kappa_{n-1}$ if $n > 0$. The following facts are analogues of the basic properties of the forcing in [9].

Fact 4.4. (1) $\mathbb{R}$ is μ-c.c., and hence preserves all cardinals $\geq \mu$.

(2) $\mathbb{R}$ has the Prikry property: if $p \in \mathbb{R}$ and σ is a sentence in the forcing language, then there is $q \leq^* p$ which decides σ, i.e., forces σ or $\neg\sigma$.
(3) $\mathbb{R}$ adds no bounded subsets of κ.
(4) For any $\langle Y_\xi : \xi < \kappa \rangle$, a sequence of sets with $Y \in U$ and $Y_\xi \in U_\xi$ for all ξ, $x_n \in Y_{\xi_n}$ for all sufficiently large $n < \omega$.
(5) Forcing with $\mathbb{R}$ changes the cofinality of $\kappa^{+\xi}$ to ω for all $\xi < \kappa$, and therefore $\mu = \kappa^+$ in the generic extension by $\mathbb{R}$.

Proof. (1) follows from the fact that any two conditions with the same lower part are compatible, and there are fewer than μ many lower parts.

(2) For simplicity, assume that p has length 0. Partition LP into

$$B_0 = \{s \in \mathrm{LP} : \text{ there is } \vec{Y}^s \text{ such that } s^\frown \vec{Y}^s \Vdash \sigma\},$$

$$B_1 = \{s \in \mathrm{LP} : \text{ there is } \vec{Y}^s \text{ such that } s^\frown \vec{Y}^s \Vdash \neg\sigma\},$$

and $B_2 = \mathrm{LP} \setminus (B_0 \cup B_1)$.

We will define a family of LP-indexed upper parts $\vec{Y}^s$. If $s \in B_0 \cup B_1$, take $\vec{Y}^s$ to be an upper part such that $s^\frown \vec{Y}^s$ decides σ. Otherwise, let $Y^s_{\xi_s} = \{x \in X_{\xi_s} : s^\frown x \in B_2\} \in U_{\xi_s}$ for all ξ and $Y_\xi = X_\xi$ for $\xi_s < \xi < \kappa$. We check that if $s \in B_2$, then $Y^s_{\xi_s} \in U_{\xi_s}$, since otherwise there is $i \in \{0, 1\}$ so that $\{x \in X_{\xi_s} : s^\frown x \in B_i\} \in U_{\xi_s}$, which would imply $s \in B_i$.

Take r to be the diagonal intersection of the $\vec{Y}^s$. If the empty lower part is in $B_0 \cup B_1$, then we are done. Otherwise, assume $H = \langle x_0, x_1, \ldots \rangle$ was obtained by forcing below r, so by induction $H \restriction n \in B_2$ for all n, contradicting the genericity of H.

(3) is immediate from (2) and the $< \kappa$-closure of the $\leq^*$ ordering, and (4) is a straightforward density argument.

(5) By a density argument, $\kappa^{+\xi} = \bigcup_{n<\omega}(x_n \cap \kappa^{+\xi})$ for all $\xi < \kappa$. □

4.3. **The final model.** For each $\xi < \kappa$ and $\gamma < \kappa^{+\xi+1}$, let $F^\gamma_\xi : X_\xi \to \kappa$ be a function representing γ in the ultrapower by U_ξ with $F^\gamma_\xi(y) < \kappa_y^{\xi+1}$ for all $y \in X_\xi$. Define a sequence of functions $\langle f_\alpha : \alpha < \mu \rangle$ of $\prod_{n<\omega} \kappa_n^{+\xi_n+1}$ by

$$f_\alpha(n) = F^{G_\alpha(\xi_n)}_{\xi_n}(x_n).$$

Following Cummings–Foreman [2], we prove the following claim:

Claim 4.5. *In $V[G * H]$, the sequence $\langle f_\alpha : \alpha < \mu \rangle$ is a scale on $\prod_{n<\omega} \kappa_n^{+\xi_n+1}$.*

Proof of Claim 4.5. It is easy to see that $\langle f_\alpha : \alpha < \mu \rangle$ is $<^*$-increasing.

Suppose $g \in \prod_{n<\omega} \kappa_n^{+\xi_n+1}$. Working in $V[G]$, let $\dot{g}$ be a $\mathbb{R}$-name for g, and let $p \in \mathbb{R}$ be arbitrary. We will find $q \leq p$ and $\alpha < \mu$ such that $q \Vdash g <^* f_\alpha$.

For simplicity, assume that p is the trivial condition and forces $\dot{g} \in \prod_{n<\omega} \kappa_n^{+\xi_n+1}$ (otherwise, we would just work below such a condition extending p). A lower part t of length $n+1$ determines the value of $\kappa_n < \kappa$, hence using the Prikry property, we can find an upper part $\vec{Y}^t$ such that $t^\frown \vec{Y}^t$ determines the value of $\dot{g}(n)$, and call this value $h(t)$. Let q be the element of $\mathbb{R}$ with empty lower part and upper part equal to the diagonal intersection of the family $\langle \vec{Y}^t \rangle$, $t \in \mathrm{LP}$, so any condition of length $n+1$ compatible with q with lower part t determines the value of $\dot{g}(n)$ as $h(t)$.

For each $\xi < \kappa$ and $x \in X_\xi$, let

$$H_\xi(x) = \sup\{h(t) + 1 : t \text{ is a lower part with last coordinate } x\}.$$

Subclaim. For each $\xi < \kappa$, H_ξ represents an ordinal γ_ξ which is less than $\kappa^{+\xi+1}$ in the ultrapower by U_ξ.

It suffices to show that for any $\xi < \kappa$ and U_ξ-almost every $x \in X_\xi$, there are fewer than $\kappa_x^{+\xi+1}$ many lower parts with last coordinate x, and therefore $H_\xi(x) < \kappa_x^{+\xi+1}$. First note that $\{x \in X_\xi : (\forall \zeta \leq \xi)\,(\kappa_x^{+\zeta+1})^{<\kappa_x} = \kappa_x^{+\zeta+1}\} \in U_\xi$ by a reflection argument since the GCH holds above κ in V and $\mathbb{P}$ does not add new sets of ordinals of size $< \kappa$. Now suppose x is in this set, and $y \in X_\zeta$ appears before x in some lower part, so $\zeta < \xi$. Then y is a subset of $x \cap \kappa_x^{+\zeta+1}$, which has order-type $\kappa_x^{+\zeta+1}$ by the definition of X_ξ. The number of subsets of $\kappa_x^{+\zeta+1}$ of size $\kappa_y^{+\zeta+1}$ is equal to $\kappa_x^{+\zeta+1} < \kappa_x^{+\xi+1}$, proving the subclaim.

Since $\langle G_\alpha : \alpha < \mu\rangle$ is a scale, there is $\alpha < \mu$ such that $\gamma_\xi < G_\alpha(\xi)$ for large ξ. Therefore, $B_\xi := \{x \in X_\xi : H_\xi(x) < F_\xi^{G_\alpha(\xi)}(x)\} \in U_\xi$ for large enough ξ. Let $H = \langle x_0, x_1, \ldots\rangle$ be the $\mathbb{R}$-generic sequence obtained by forcing below q. Using Fact 4.4 part (4), for sufficiently large n, $x_n \in B_{\xi_n}$ and therefore:

$$g(n) = h(H\restriction n+1) < H_\xi(x_n) < F_{\xi_n}^{G_\alpha(\xi_n)}(x_n) = f_\alpha(n).$$

□

Claim 4.6. *In $V[G * H]$, the scale $\langle f_\alpha : \alpha < \mu\rangle$ is continuous.*

Proof of Claim 4.6. Let $\beta < \mu$ be a limit ordinal. We will check that f_β is an exact upper bound for $\langle f_\alpha : \alpha < \beta\rangle$. We can assume that $\omega < \mathrm{cf}(\beta)^V < \kappa$, since all other points have cofinality ω in $V[G * H]$. Working in $V[G]$ and using that $\langle G_\alpha : \alpha < \mu\rangle$ is a good scale, we can find A unbounded in β with order-type $\mathrm{cf}(\beta)$ and some $\xi_0 < \omega$ so that $\langle G_\alpha(\xi) : \alpha \in A\rangle$ is strictly increasing for each $\xi > \xi_0$. Therefore, $\xi \mapsto \sup\{G_\alpha(\xi) : \alpha \in A\}$ is an exact upper bound for $\langle G_\alpha : \alpha < \beta\rangle$. Using continuity of $\langle G_\alpha : \alpha < \mu\rangle$, we can pick ξ_0 large enough so that $G_\beta(\xi) = \sup\{G_\alpha(\xi) : \alpha \in A\}$ for all $\xi > \xi_0$. For each $\xi > \xi_0$,

$$\{x \in X_\xi : \langle F_\xi^{G_\alpha(\xi)}(x) : \alpha \in A\rangle \text{ is increasing with supremum } F_\xi^{G_\beta(\xi)}(x)\} \in U_\xi.$$

In $V[G*H]$, Fact 4.4 part (4) then implies that there is $n_0 < \omega$ so that for every $n \geq n_0$, $\langle f_\alpha(n) : \alpha \in A\rangle$ is strictly increasing with supremum $f_\beta(n)$. For any $h < f_\beta$ and any $n_0 < n < \omega$, there is $\alpha_n \in A$ so that $h(n) < f_{\alpha_n}(n)$. Let $\alpha^* < \beta$ be greater than $\sup_n \alpha_n$. Then $h <^* f_{\alpha^*}$, so f_β is an exact upper bound for $\langle f_\alpha : \alpha < \beta\rangle$. □

It remains to check that every $A \subseteq \mu$ is careful in $V[G * H]$. Working in $V[G]$, let $\dot{A}$ be a $\mathbb{R}$-name for A. For each $s \in \mathrm{LP}$, let

$$A_s = \{\alpha \in \mu : \text{ there exists an upper part } \vec{Y}^s \text{ such that } s^\frown\vec{Y}^s \Vdash \alpha \in \dot{A}\}.$$

By 4.1, there is in $V[G]$ a function $E : \mathrm{LP} \times \mu \to \kappa$ such that:

(1) If (s,α) and (s,β) are in $\mathrm{dom}(E)$ with $\alpha \in A_s$ and $\beta \notin A_s$, then $d_G^*(\alpha,\beta) \leq \max\{E(s,\alpha), E(s,\beta)\}$.
(2) If $(t,\alpha) \in \mathrm{dom}(E)$, $s \subseteq t$, and $\alpha \in A_u$ for all $s \subseteq u \subseteq t$, then $E(t,\alpha) = E(s,\alpha)$.
(3) If $(t,\alpha) \in \mathrm{dom}(E)$, $s \subseteq t$, and $\alpha \notin A_u$ for all $s \subseteq u \subseteq t$, then $E(t,\alpha) = E(s,\alpha)$.

In $V[G * H]$, we will find a function F as in Lemma 2.13 which shows that A is careful. For any given $\alpha < \mu$, there is $p \in \mathbb{R}$ that either forces $\alpha \in \dot{A}$ or $\alpha \notin \dot{A}$. If s is the lower part of p, then set $F_0(\alpha)$ to be the length of s. Now either

$$\alpha \in A_t \text{ for all } t \subseteq H \text{ extending } s,$$

or

$$\alpha \notin A_t \text{ for all } t \subseteq H \text{ extending } s,$$

where the measure 1 sets that witness membership (or nonmembership) in A_t come from the upper part of s. In either case, the value of $E(t, \alpha)$ is constant for $s \subseteq t \subseteq H$, and let $F_1(\alpha)$ be the least n such that ξ_n is greater than this constant value.

For each $t \in \mathrm{LP}$, $\xi < \kappa$, define $S^t_\xi = \{G_\beta(\xi) : \beta \in A_t \text{ and } E(t, \beta) \leq \xi\}$ (the sequence $\langle S^t_\xi : \xi < \kappa \rangle$ witnesses that A_t is careful on the scale $\langle G_\alpha : \alpha < \mu \rangle$). Then S^t_ξ is a subset of $\kappa^{+\xi+1}$, and since the U_ξ ultrapower is closed under $\kappa^{+\xi+1}$ sequences, S^t_ξ is a member of this ultrapower, and hence is represented in the ultrapower by a function s^t_ξ with domain X_ξ.

If $\alpha \in A_t$, then define

$$Y^{\alpha,t}_\xi = \{y \in X_\xi : F^{G_\alpha(\xi)}_\xi(y) \in s^t_\xi(y)\}$$

if $\xi \geq E(t, \alpha)$, and X_ξ otherwise. Then $Y^{\alpha,t}_\xi \in U_\xi$ for each $\xi < \kappa$, since $G_\alpha(\xi) \in S^t_\xi$ in the first case of the definition and it is trivial in the second. If $\alpha \notin A_t$, define $Y^{\alpha,t}_\xi = \{y \in X_\xi : F^{G_\alpha(\xi)}_\xi(y) \notin s^t_\xi(y)\}$ if $\xi \geq E(t, \alpha)$, and X_ξ otherwise. In the first case of the definition, property (1) of E guarantees that $G_\alpha(\xi) \notin S^t_\xi$, so again $Y^{\alpha,t}_\xi \in U_\xi$ for each $\xi < \kappa$.

Let $\langle Y^\alpha_\xi : \xi < \kappa \rangle$ be the diagonal intersection of the $Y^{\alpha,t}_\xi$. By Fact 4.4 part (4), there is $N < \omega$ such that $x_n \in Y_{\xi_n}$ for all $n \geq N$; let $F_2(\alpha)$ be such an N.

Finally, define $F(\alpha)$ to be the maximum of $F_0(\alpha), F_1(\alpha), F_2(\alpha)$. Now if $\alpha \in A$ and $\beta \notin A$, we must verify that $d^*(\alpha, \beta) < \max\{F(\alpha), F(\beta)\}$ (here d^* denotes the d^* function for the scale $\langle f_\alpha : \alpha < \mu \rangle$). In other words, we must show that for any $n \geq \max\{F(\alpha), F(\beta)\}$, $f_\alpha(n) \neq f_\beta(n)$. Let $t = \langle x_0, x_1, \ldots, x_{n-1} \rangle$ be the initial segment of H of length n. Since $n \geq F_0(\alpha), F_0(\beta)$, $\alpha \in A_t$ and $\beta \notin A_t$. Since $n \geq F_2(\alpha), F_2(\beta)$, $x_n \in Y^\alpha_{\xi_n} \cap Y^\beta_{\xi_n}$, and since $n \geq F_1(\alpha), F_1(\beta)$ both $Y^\alpha_{\xi_n}$ and $Y^\beta_{\xi_n}$ were defined using the first cases of their respective definitions. Therefore,

$$f_\alpha(n) = F^{G_\alpha(\xi_n)}_{\xi_n}(x_n) \in s^t_\xi(x_n) \quad \text{and} \quad f_\beta(n) = F^{G_\beta(\xi_n)}_{\xi_n}(x_n) \notin s^t_\xi(x_n)$$

and hence $f_\alpha(n) \neq f_\beta(n)$.

4.4. **Making κ into the least cardinal fixed point.** Using techniques originating in Magidor [10], collapses can be interleaved into the forcing of Theorem 3 so that κ becomes the least cardinal with $\kappa = \aleph_\kappa$ in the final model. (In [9], interleaving collapses in diagonal Prikry forcing was used to turn κ into $\aleph_{\omega^2}$). We will roughly sketch this construction. Working in $V[G]$, for each $\xi < \kappa$ let $i_\xi : V[G] \to N_\xi$ be the ultrapower by U_ξ. In N_ξ, $\mathrm{Col}(\kappa^{+\kappa+2}, i_\xi(\kappa))^{N_\xi}$ has cardinality $i_\xi(\kappa)$ and $i_\xi(\kappa)$-c.c. Back in $V[G]$, $|i_\xi(\kappa)| \leq \kappa^{\kappa^{\xi+1}} = \kappa^{\xi+2}$, so using the $< \kappa^{\xi+2}$-closure of the poset and N_ξ, we can find K_ξ which is $\mathrm{Col}(\kappa^{+\kappa+2}, i_\xi(\kappa))^{N_\xi}$-generic over N_ξ. Then we can replace $\mathbb{R}$ in the construction of Theorem 3 by the forcing whose conditions are of the form

$$p = \langle c^p, x^p_0, f^p_0, \ldots, x^p_{n-1}, f^p_{n-1} \rangle^\frown \langle Y_\xi, F_\xi : \xi < \kappa \rangle$$

where

- $\langle x_0, \ldots, x_{n-1}\rangle^\frown\langle Y_\xi : \xi < \kappa\rangle$ is a condition from the diagonal Prikry forcing defined above.
- $c \in \mathrm{Col}(\omega, < \kappa_0)$.
- For all $i < n-1$, $f_i \in \mathrm{Col}(\kappa_i^{+\kappa_i+2}, < \kappa_{i+1})$.
- $f_{n-1} \in \mathrm{Col}(\kappa_{n-1}^{+\kappa_{n-1}+2}, \kappa)$.
- For $\xi \geq \kappa_{n-1}$, F_ξ is a function with domain Y_ξ such that $F_\xi(x) \in \mathrm{Col}(\kappa_x^{\kappa_x+2}, \kappa)$ and F_ξ represents an element of K_ξ in the U_ξ ultrapower.

A condition $q = \langle c^q, x_0^q, f_0^q, \ldots, x_{m-1}^q, f_{m-1}^q\rangle^\frown\langle Y_\xi^q, F_\xi^q : \xi < \kappa\rangle$ extends p if

- $\langle x_0^q, \ldots, x_{n-1}^q\rangle^\frown\langle Y_\xi^q : \xi < \kappa\rangle \leq \langle x_0^p, \ldots, x_{n-1}^p\rangle^\frown\langle Y_\xi^p : \xi < \kappa\rangle$ as conditions from $\mathbb{R}$.
- $c^q \leq c^p$ and $f_i^q \leq f_i^p$ for all $i < n$.
- For all $n \geq i < m$, $f_i^q \leq F_i^p(x_i^q)$.
- For all $\xi \geq \kappa_{m-1}$ and all $x \in Y_\xi^q$, $F_\xi^q(x) \leq F_\xi^p(x)$.

The restriction on the F_ξ is needed to prove the μ-c.c. and the Prikry property.

In the extension, if $\eta < \kappa$, then $\eta < \kappa_n$ for some n, and therefore $\aleph_\eta < \aleph_{\kappa_n} \leq \kappa_{n+1} < \kappa$, so $\kappa = \aleph_\kappa$. Furthermore, κ_0 is collapsed to be ω_1 and for $n > 0$, κ_n is the nth iteration of the map $\eta \mapsto \eta^{+\eta+3}$ evaluated at κ_0, so κ must be the least cardinal fixed point.

5. A stationary A with $\nu(A) = \emptyset$

Suppose that $B \subseteq \kappa^+$ is careful. Let $\vec{T} = \langle T_n : n < \omega\rangle$ be any careful sequence for B, and $\vec{T}^* = \langle T_n^* : n < \omega\rangle$ be any sequence with $\mu(\vec{T}^*) = B$. Then $\nu(\langle T_n \setminus T_n^*\rangle) = \emptyset$, that is, $\vec{T}$ is the minimum sequence with $\mu(\vec{T}) = B$, up to sequences which evaluate to $\emptyset$ under ν. We would like $\vec{T}$ to be the minimum such sequence modulo nonstationary sequences. This is equivalent to asking that any $\vec{S}$ with $\nu(\vec{S}) = \emptyset$ has S_n nonstationary for all but finitely many n.

Investigating this question from another angle, notice that

- The unboundedness property of a scale says that for any unbounded $A \subseteq \kappa^+$ and $\vec{S}$ with $\mu(\vec{S}) = A$, S_n is unbounded in κ_n for all but finitely many n.
- Proposition 2.6 says that for stationary A, the S_n must be stationary for all but finitely many n.

The original question can be rephrased: if A is club (or even equal to the whole of κ^+) and $\mu(\vec{S}) = A$, must S_n be club in κ_n for all but finitely many n? We will give a negative answer under the assumption of a continuous tree-like scale. This result, and its proof, are similar to Theorem 3 of [6].

Theorem 4. *Let $\langle f_\alpha : \alpha < \kappa^+\rangle$ be continuous and tree-like, and let $\eta < \kappa_0$ be a regular cardinal. There is a sequence $\langle S_n : n < \omega\rangle$, $S_n \subseteq \kappa_n$, such that $\nu(\vec{S}) = \emptyset$ (equivalently $\mu(\langle \kappa_n \setminus S_n\rangle) = \kappa^+$) and S_n is stationary in $\kappa_n \cap \mathrm{Cof}(\eta)$ for all n.*

Proof. Consider the tree $\mathcal{T}$ of initial segments of members of $\vec{f}$. So a node on level n is a sequence of ordinals of length n where the mth term is $< \kappa_m$, but because $\vec{f}$ is tree-like there is no ambiguity to identify the node by its last (i.e., $(n-1)$st) term, so if $\beta \in \kappa_{n-1}$ is a node on the nth level of the tree, let $s_n(\beta)$ be sequence identified with it. Let $<_{\mathcal{T}}$ denote the tree order.

The main point of the proof is that this tree can be thinned to be stationarily branching after some point, a general fact about continuous tree-like scales which may be of independent interest.

Lemma 5.1. *There is a subtree $\mathcal{T}'$ of $\mathcal{T}$ such that there is some $\gamma \in \mathcal{T}'$ compatible with every element of $\mathcal{T}'$ (γ is called the stem), and for every α on the nth level of $\mathcal{T}'$, where $n \geq$ level(γ), the set $\{\beta \in \kappa_n : \alpha <_{\mathcal{T}'} \beta\} \cap \mathrm{Cof}(\eta)$ is stationary.*

Proof. We first define a game. On the nth turn, player I plays $A_n \subset \kappa_n$ nonstationary and player II plays $\alpha_n \in \kappa_n$. In addition player II plays $N \in \omega$ on the 0th turn. Player II wins if $\alpha_n <_{\mathcal{T}} \alpha_{n+1}$ for all n and $\alpha_n \in \mathrm{Cof}(\eta) \setminus A_n$ for all $n > N$. Otherwise player I wins.

We will show that II has a winning strategy. This game is open, hence determined, so towards a contradiction assume that I has a winning strategy σ. Let $M \prec (H(\kappa^+), \sigma)$, where M is internally approachable of length η. Then $\chi_M =^* f_\alpha$ for some $\alpha \in \kappa^+$, and set $\alpha_n = f_\alpha(n)$. Choose N so that $\chi_M(n) = f_\alpha(n)$ has cofinality η for all $n > N$. We show that II can play the α_n and N against σ and win, a contradiction. For each $n > N$, let $B_n = \bigcup_{\beta \in \kappa_{n-1}} \sigma(s_{n-1}(\beta))$ be the union of all possible plays of I according to σ, where the union ranges over all β on the nth level of $\mathcal{T}$. Each $\sigma(s)$ is a nonstationary subset of κ_n, so this union is nonstationary in κ_n. Furthermore, since $\sigma \in M$, we have that $B_n \in M$, so its complement is a club C_n in κ_n which is a member of M. Therefore $\alpha_n \in C_n$ for all $n > N$. By the definition of C_n, $\alpha_n \notin \sigma(s_{n-1}(\alpha_{n-1}))$.

Let τ be a winning strategy for II. We may assume that II's 0th move according to τ does not depend on I's, since by the definition of the game, I's 0th move is meaningless. So let N be the 0th move that II plays. We may also assume that II's first N moves according to τ do not depend on I's moves. Then the subtree of plays according to τ in $\mathcal{T}$ is stationarily branching in cofinality η with stem of length N, since otherwise at a nonstationarily branching play, I could block by playing all successors. This completes the proof of the lemma. $\square$

Now we fix ordinals $\langle \beta_n : N < n < \omega \rangle$ such that $\beta_n \in \mathrm{Cof}(\eta)$ is on the $(n-1)$st level of $\mathcal{T}'$ and the β_n form an antichain in $\mathcal{T}'$. Then let $S_n \subseteq \kappa_n$ be the successors of β_n for each $n < \omega$. By stationary branching, S_n is stationary in $\kappa_n \cap \mathrm{Cof}(\eta)$. Since the scale is tree-like, for any $\alpha \in \kappa^+$ there is at most one n such that $f_a(n) \in S_n$, so $\nu(\vec{S}) = \emptyset$. Thus, the theorem is proved. $\square$

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