Fraction conversion

Given base b, say b =3.

What do we mean by a base 3 decimal 0.1211_3?

Each digit stands for a multiple of a negative power of 3

$$0.1021_3 = \frac{1}{3} + 0 \times \frac{1}{3^2} + 2 \times \frac{1}{3^3} + 1 \times \frac{1}{3^4}$$

$$= \frac{1}{3} + \frac{2}{27} + \frac{1}{81} \approx 0.41975308 \cdots$$

Example - Hex fraction to decimal

Suppose base b = 16.
What is the value of decimal expression of the hex expression 0.A3E ?

As before

$$0.A3E_H = A \times \frac{1}{16} + 3 \times \frac{1}{16^2} + E \times \frac{1}{16^3}$$
$$= \frac{10}{16} + \frac{3}{256} + \frac{14}{4096}$$
$$= \frac{2560 + 48 + 14}{4096} \approx 0.640136 \cdots$$

YORK U

Previous examples show conversion to a decimal fraction from a fraction with a base other than 10 is easy.
Other direction - more difficult

Example: convert the decimal fraction 0.381 to a base 8 binary fraction.

We know 0.381 indicates a sum of multiples of negative powers of 10

$$0.381 = \frac{3}{10} + \frac{7}{10^2} + \frac{6}{10^3}$$

We need to express it as a sum of negative powers of 8

Expressing 0.381 as sum of negative powers of 8

Some negative powers of 8 are:

$$\frac{1}{8} = 0.125, \quad \frac{1}{8^2} = \frac{1}{64} = .015625, \quad \frac{1}{8^3} = \frac{1}{512} = 0.001953125$$

Find greatest multiple of $1/8$ less than or equal to 0.381.

Next find greatest multiple of $1/8^2$ $\leq$ remainder ----- and so on.

Observe that : $\frac{3}{8} = 0.375 < 0.381$ and $0.381 = \frac{3}{8} + 0.006$

Thus first digit of base 8 fractional expression is 3

Now for remainder 0.006 find the least multiple of $\frac{1}{8^2}$ such that $\frac{1}{8^2} \leq 0.006$

Note however that: $0.006 < \frac{1}{8^2} = 0.015625$

So we can find no such multiple of $\frac{1}{8^2}$

Thus 2nd digit of the base 8 expansion must be 0

To find 3rd digit look we now look for the least multiple of

$$\frac{1}{8^3} = 0.001953125$$

that is less than or equal to 0.006

Note:

$$\frac{3}{8^3} = 0.0058569375 < 0.006 < \frac{4}{8^3} = 0.0078125$$

So 3 is the least multiple and then equals the 3rd digit

To continue and find the 4th digit we need to find the least multiple of $\frac{1}{8^4} = \frac{1}{4086} = 0.000244140625$

That is less than or equal to the difference

$$0.006 - 0.0058569375 = 0.0001430625$$

I think we can agree that this is a messy process

An alternative method is needed

Alternate method for fraction conversion

Alternate method for number conversion involved dividing successively by the value of the base. Result: the decimal point is moved to the left and digits drop out as remainders

For decimal conversion move decimal point to left by multiplying by the value of the base

Example: Convert 0.828125 to binary

Observe: $(0.828125)\times 2 = 1.656250$

so that: $0.828128 = \frac{1}{2} + \frac{0.656250}{2} = 0.1_2 + \frac{0.656250}{2}$

and the first binary digit is 0.1

The remaining binary digits are buried in 0.656250

Repeat the process

$(0.656250)\times 2 = 1.31250$

getting $0.656250 = \frac{1}{2} + \frac{0.31250}{2}$

so that: $0.828125 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2} 0.31250 = 0.11_2 + \frac{0.31250}{2}$

continuing from : $$0.828125 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2}0.31250$$

$$(0.31250) \times 2 = 0.62500 \Rightarrow 0.31250 = \frac{0.62500}{2}$$

so that

$$0.828125 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2}\left(\frac{0.62500}{2}\right) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}0.6250$$

Continuing

$$(0.6250) \times 2 = 1.250 \Rightarrow 0.6250 = \frac{1.250}{2} = \frac{1}{2} + \frac{0.250}{2}$$

so that

$$0.828125 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\left(\frac{1}{2} + \frac{0.250}{2}\right)$$

or

$$0.828125 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{0.250}{2^4} = 0.1101_2 + \frac{0.250}{2^4}$$

Example: Convert 0.78125 to hex

As before multiply by the base to shift the decimal point

$$0.78125 \times 16 = 12.5 \Rightarrow 0.7825 = \frac{12}{16} + \frac{0.5}{16} = C_H + \frac{0.5}{16}$$

$$0.5 \times 16 = 8 \Rightarrow 0.5 = \frac{8}{16}$$

$$0.78125 = \frac{12}{16} + \frac{8}{16^2} = C8_H$$

Remark

Converting fractions as multiples of negative powers of one base to fractions as multiples of negative powers of another base may be non-ending

Example:

1/3 expressed as multiples of negative powers of 10 is

0.3333333 ......

1/3 expressed as multiples of negative powers of 3 gives expression

0.1

Binary Addition - essence of calculation

A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	10

"two"

Hint: Learn This Table !!

Compare to the bit-wise XOR logic operation applied to two bit streams.

For each bit - A or B - of two bit streams of equal length apply

A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0

Easily implemented - needs to account for "carrying" 1 when A = 1 and B = 1

YORK U

Examples

```
 1       1
   1 0 1 0 1        21
 + 1 1 0 0 1       +25
 -----------       ---
 1 0 1 1 1 0        46

  1  1    1    1   1
     1   1   0   1   1   0   1
                 1   0   1   1   0
 ---------------------------------
   1 0 0   0   0   0   1   1
```

Observation:

Multiplication by a power of base b^k shifts decimal point .

If k > 0 shift to right k positions
If k < 0 shift to left k positions

Binary multiplication

Observe 2^k in binary is 1 followed by k zeros

We know that multiplying a binary number 2^k has the effect of moving the decimal k places to the right

But 2^k is simply 1 followed by k zeros - thus the following

$$
\begin{array}{r}
101\ 110\ 1.1 \\
\times 1000 \\
\hline
10\ 1110\ 110\ 0.0
\end{array}
$$

Application:

Observe $1101 \times 101 = 1101 \times (100+1)$

$$= (1101 \times 100) + (1101 \times 1)$$

$$= 110100 + 1101$$

Performing the addition gives:

$$\begin{array}{r} 110100 \\ +\ 1101 \\ \hline 1000001 \end{array}$$

Now use our grade 4 method adapted to binary.

Note: the method implements shifting

$$\begin{array}{r} 1101 \\ \times 101 \\ \hline 1101 \\ 0 \\ +\ 1101 \\ \hline 1000001 \end{array}$$