# Adaptive Kalman Filtering
## Based on Posteriori Variance-Covariance Components Estimation

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**ABSTRACT**

There are different ways to construct adaptive Kalman filtering (AKF) algorithms. This paper proposes an innovative way to simultaneously estimate the variance matrix $\boldsymbol{R}$ of the measurement vector and the variance matrix $\boldsymbol{Q}$ of the process noise vector based on the variance-covariance component estimation by taking the advantages of the measurement residuals and the process noise residuals (Wang, 1997, 2009; Wang et al, 2009) and the measurement redundancy contribution (Ou, 1989). The core of the novel AKF algorithm lies in the projection of the system innovation vector into the three groups of residuals: the residuals of the measurement vector, the residuals of the process noise vector and the residuals of the predicted state vector exclusive of the effect of the process noise. The simulated and real GPS data in kinematic relative positioning mode were used to demonstrate the performance of the proposed adaptive Kalman filter. The results from the simulated datasets confirm to the simulated variance-covariance components well. The results from real kinematic GPS datasets are also provided and discussed.

**Key words:** Adaptive Kalman Filter, measurement residuals, process noise residuals, variance-covariance components, kinematic GPS, relative positioning, redundancy contribution.

## 1. INTRODUCTION

Kalman filter (KF) has become one of the most widely used recursive methods that estimate the states of a process. If the system and measurement model, the variance-covariance (VC) matrix $\boldsymbol{Q}$ of the process noise vector and the VC matrix $\boldsymbol{R}$ of the measurement vector are known, the KF produces an optimal solution. In practice, one cannot guarantee to have them all satisfied. In particular, the a-priori $\boldsymbol{Q}$ and $\boldsymbol{R}$ are unknown or approximated by applying the best available knowledge. This may produce unreliable results or cause the KF to diverge. The classical method that chooses the values for these two matrices is through empirical analysis on the system and measurement errors or manually in an ad hoc fashion. This requires an experienced developer. However, the results from empirical analysis may not quantitatively represent the real $\boldsymbol{Q}$ and $\boldsymbol{R}$. By performing the adaptive Kalman Filter (AKF), one can make the filter adaptively apply the realistic noise statistics $\boldsymbol{Q}$ and $\boldsymbol{R}$ that are based on the being processed samples.

There are different ways to construct AKF algorithms. The very pioneer work from Mehra (1970) used the autocorrelation functions of the innovation sequence to estimate both $\boldsymbol{Q}$ and $\boldsymbol{R}$ for the time invariant system. Another popular alternate method was to introduce the adaptive algorithms based on the system innovation sequence either to derive $\boldsymbol{R}$ and a scale factor for $\boldsymbol{Q}$ (Ding et al, 2007), or to indirectly estimate $\boldsymbol{Q}$ and $\boldsymbol{R}$ (Mohamed and Schwarz, 1999). Hu et al (2003) assumed $\boldsymbol{R}$ completely known in order to estimate $\boldsymbol{Q}$. Yang and Gao (2006) derived two optimal adaptive factors, based on either the estimated innovation VC matrix or the estimated VC of the predicted state vector, to balance the contribution of the dynamic system and the measurement model information.

In this work, the authors propose an alternate way to simultaneously estimate $\boldsymbol{R}$ and $\boldsymbol{Q}$ by performing the variance-covariance component estimation using the measurement residual vector and the process noise residual vector that were derived in (Wang, 2009) and the measurement redundancy index formulated by (Ou, 1989). After the overview of the Kalman filtering and adaptive Kalman filtering in Section 2, the proposed AKF algorithm is presented in Section 3. To illustrate the proposed adaptive Kalman filter, the kinematic relative GPS positioning is taken as an example (Section 3.4). Both of the simulated and real GPS data were processed. Our numerical analysis of the proposed algorithm limits $\boldsymbol{Q}$ to be diagonal and $R$ to be correlated. Section 4 provides the corresponding results along with specific analysis. Remarks and conclusions are provided in Section 5.

## 2. ADAPTIVE KALMAN FILTERING

### 2.1. OVERVIEW OF KALMAN FILTER

With provision for algorithm formulation, a concise summary of Kalman filter is given in this section. A linear

or linearized multivariable discrete system is considered over a discrete time series $\{t_0, t_1, ..., t_k, \ldots, t_N\}$, often simplified to $\{0, 1, \ldots, k, ..., N\}$. Without loss of generality, the deterministic system input is intentionally omitted here. Straightforward, the system can be described at instant $k$ as follows:

$$\boldsymbol{x}(k) = \boldsymbol{A}(k-1)\boldsymbol{x}(k-1) + \boldsymbol{B}(k-1)\boldsymbol{w}(k-1) \quad (1)$$

$$\boldsymbol{z}(k) = \boldsymbol{C}(k)\boldsymbol{x}(k) + \Delta(k) \quad (2)$$

wherein $\boldsymbol{x}(k)$ is the $n\times 1$ state-vector; $\boldsymbol{z}(k)$ is the $p\times 1$ observation vector, $\boldsymbol{w}(k-1)$ is the $m\times 1$ process noise vector, $\Delta(k)$ is the $p\times 1$ measurement noise vector; $\boldsymbol{A}(k-1)$ is the $n\times n$ transition matrix; $\boldsymbol{B}(k-1)$ is the $n\times m$ coefficient matrix of $\boldsymbol{w}(k-1)$; $\boldsymbol{C}(k)$ is the $p\times n$ output matrix. The random vectors $\boldsymbol{w}(k-1)$ and $\Delta(k)$ are generally assumed to be: $\boldsymbol{w}(k-1) \sim \boldsymbol{N}(\boldsymbol{o}, \boldsymbol{Q}(k-1))$ and $\Delta(k) \sim \boldsymbol{N}(\boldsymbol{o}, \boldsymbol{R}(k))$ with zero-means $\boldsymbol{o}$ and the variance matrices as $\boldsymbol{Q}(k-1)$ and $\boldsymbol{R}(k)$ positive definite, respectively. Further assumptions about $\boldsymbol{w}(\cdot)$ and $\Delta(\cdot)$ are considered: $\boldsymbol{Cov}(\boldsymbol{w}(i), \boldsymbol{w}(j)) = \boldsymbol{O}$, $\boldsymbol{Cov}(\Delta(i), \Delta(j)) = \boldsymbol{O}$ and $\boldsymbol{Cov}(\boldsymbol{w}(i), \Delta(j)) = \boldsymbol{O}$ for $i \neq j$. Commonly, one also assumes to have the initial state vector $\boldsymbol{x}(0)$ and its variance-covariance matrix $\boldsymbol{D}_{xx}(0)$ available and independent of arbitrary $\boldsymbol{w}(\cdot)$ and $\Delta(\cdot)$. Under the given assumptions, the unbiased optimal estimation of $\boldsymbol{x}(k)$ can be derived in the sense of minimum variance an in table 1.

Table 1: Solution of Kalman filter

| |
|---|
| The predicted state vector and its variance matrix<br>$\hat{\boldsymbol{x}}(k/k-1) = \boldsymbol{A}(k-1)\hat{\boldsymbol{x}}(k-1)$<br>$\boldsymbol{D}_{xx}(k/k-1) = \boldsymbol{A}(k-1)\boldsymbol{D}_{xx}(k-1)\boldsymbol{A}^T(k-1) + \boldsymbol{B}(k-1)\boldsymbol{Q}(k-1)\boldsymbol{B}^T(k-1)$ |
| The optimal estimated state vector and its variance matrix<br>$\hat{\boldsymbol{x}}(k) = \hat{\boldsymbol{x}}(k/k-1) + \boldsymbol{G}(k)\boldsymbol{d}(k)$<br>$\boldsymbol{D}_{xx}(k) = \boldsymbol{G}(k)\boldsymbol{R}(k)\boldsymbol{G}^T(k) + [\boldsymbol{E} - \boldsymbol{G}(k)\boldsymbol{C}(k)]\boldsymbol{D}_{xx}(k/k-1)[\boldsymbol{E} - \boldsymbol{G}(k)\boldsymbol{C}(k)]$ |
| The optimal estimated state vector and its variance matrix<br>$\boldsymbol{d}(k) = \boldsymbol{z}(k) - \boldsymbol{C}(k)\hat{\boldsymbol{x}}(k/k-1)$<br>$\boldsymbol{D}_{dd}(k) = \boldsymbol{C}(k)\boldsymbol{D}_{xx}(k/k-1)\boldsymbol{C}^T(k) + \boldsymbol{R}(k)$ |
| The gain matrix<br>$\boldsymbol{K}(k) = \boldsymbol{D}_{xx}(k/k-1)\boldsymbol{C}^T(k)\boldsymbol{D}_{dd}^{-1}(k)$ |

### 2.2. ADAPTIVE KALMAN FILTERING

As a fact, the effect of the initial state vector along with its variance will be forgotten with the time being in Kalman filtering. However, a major obstacle in applying Kalman filter is specifying the variance matrices $\boldsymbol{Q}(\cdot)$ and $\boldsymbol{R}(\cdot)$ (Louv, 1984 etc.). Their true values are not known. Users make their good effort to approximate $\boldsymbol{Q}(\cdot)$ and $\boldsymbol{R}(\cdot)$ using the best available information about their applications. In general, the specified values for the variance matrices $\boldsymbol{Q}(\cdot)$ and $\boldsymbol{R}(\cdot)$ are experimental.

Primarily, two different strategies for running adaptive Kalman filtering have been developed. The majority of the adaptive algorithms focused on how to sequentially improve $\boldsymbol{Q}(\cdot)$ or $\boldsymbol{R}(\cdot)$, or both of $\boldsymbol{Q}(\cdot)$ and $\boldsymbol{R}(\cdot)$ (Jazwinski, 1969; Mehra, 1970; Louv, 1984; Mohamed and Schwarz, 1999; Hu and Liu, 2002; etc.). Another strategy aims to find a balance between the time update and the measurement update (Wang, 1997; Yang and Gao, 2006; ), i.e.

$$\boldsymbol{v}_{l_z}^T(k)\boldsymbol{R}(k)\boldsymbol{v}_{l_z}(k) + \alpha_k\{\delta\boldsymbol{x}^T(k)\boldsymbol{D}^{-1}(k/k-1)\delta\boldsymbol{x}(k)\} = \min \quad (3)$$

with $\delta\boldsymbol{x}(k) = \boldsymbol{x}(k) - \boldsymbol{x}(k/k-1)$.

Further derivative algorithms similar to (3) were also constructed (Ding, et al, 2007; Yang, et al, 2001; Oussalah and Schutter, 2000 etc.). This is not more adaptive than robust because the formulation is modified in addition to providing the adaptive $\boldsymbol{Q}(\cdot)$ or $\boldsymbol{R}(\cdot)$.

In this manuscript, the proposed adaptive algorithm only deals with improving of $\boldsymbol{Q}(\cdot)$ and $\boldsymbol{R}(\cdot)$ in Kalman filtering.

## 3. VARIANCE-COVARIANCE ESTIMATION-BASED ADAPTIVE KALMAN FILTERING

This section furnishes the novel adaptive algorithm in Kalman filtering in details. First, the residual vectors for the measurement vector and the process noise vector are derived. Then, the redundancy contribution in Kalman filtering is discussed. At the end of this section, the posteriori estimation of the varance and covariance factors is formulated on the ground of the measurement or pseudo-measurement residuals, not the system innovation residuals instead as usual.

### 3.1. MEASUREMENT AND PROCESS NOISE RESIDUALS

In order to derive the residuals of measurement vector and process noise vector, an alternate prospect was given about Kalman filter in (Wang, 1997, 2009; Caspary and Wang, 1998; Wang et al, 2009).

There exist three groups of stochastic information that is associated with the estimation of the state vector $\boldsymbol{x}(\boldsymbol{k})$ at the instant $\boldsymbol{k}$:

(1). the observation noise vector $\Delta(\boldsymbol{k})$,

(2). the system process noise vector $\boldsymbol{w}(\boldsymbol{k}-1)$ and

(3). the noise on the predicted state vector $\hat{\boldsymbol{x}}(\boldsymbol{k}/\boldsymbol{k}-1)$ brought by $\hat{\boldsymbol{x}}(\boldsymbol{k}-1)$ through the propagation of $\{\Delta(1),\ldots,\Delta(\boldsymbol{k}-1)\}$ and $\{\boldsymbol{w}(0),\ldots,\boldsymbol{w}(\boldsymbol{k}-2)\}$.

Customarily, "(2)" and "(3)" are considered together in the one step predicted state vector $\hat{\boldsymbol{x}}(\boldsymbol{k}/\boldsymbol{k}-1)$ from time $\boldsymbol{k}-1$ to $\boldsymbol{k}$. As a matter of fact, these three groups of stochastic information should be studied separately. By defining the independent (pseudo-)observation groups

$$\boldsymbol{l}_x(\boldsymbol{k}) = \boldsymbol{A}(\boldsymbol{k}-1)\hat{\boldsymbol{x}}(\boldsymbol{k}-1) \tag{4}$$

$$\boldsymbol{l}_w(\boldsymbol{k}) = \boldsymbol{w}_0(\boldsymbol{k}-1) \tag{5}$$

$$\boldsymbol{l}_z(\boldsymbol{k}) = \boldsymbol{z}(\boldsymbol{k}) \tag{6}$$

with their variance-covariance matrices

$$\boldsymbol{D}_{l_x l_x}(\boldsymbol{k}) = \boldsymbol{A}(\boldsymbol{k}-1)\boldsymbol{D}_{xx}(\boldsymbol{k}-1)\boldsymbol{A}^T(\boldsymbol{k}-1) \tag{7}$$

$$\boldsymbol{D}_{l_w l_w}(\boldsymbol{k}) = \boldsymbol{Q}(\boldsymbol{k}-1) \tag{8}$$

$$\boldsymbol{D}_{l_z l_z}(\boldsymbol{k}) = \boldsymbol{R}(\boldsymbol{k}) \tag{9}$$

The system model with (1) and (2) can be reformulated through the residual vectors of these three groups of the error sources[Wang, 1997; Caspary & Wang, 1998; Wang, 2008]:

$$\boldsymbol{v}_{l_x}(\boldsymbol{k}) = \hat{\boldsymbol{x}}(\boldsymbol{k}) - \boldsymbol{B}(\boldsymbol{k}-1)\hat{\boldsymbol{w}}(\boldsymbol{k}-1) - \boldsymbol{l}_x(\boldsymbol{k}) \tag{10}$$

$$\boldsymbol{v}_{l_w}(\boldsymbol{k}) = \hat{\boldsymbol{w}}(\boldsymbol{k}-1) - \boldsymbol{l}_w(\boldsymbol{k}) \tag{11}$$

$$\boldsymbol{v}_{l_z}(\boldsymbol{k}) = \boldsymbol{C}(\boldsymbol{k})\ \hat{\boldsymbol{x}}(\boldsymbol{k}) - \boldsymbol{l}_z(\boldsymbol{k}) \tag{12}$$

Here, $\boldsymbol{l}_x(\boldsymbol{k})$, $\boldsymbol{l}_w(\boldsymbol{k})$ and $\boldsymbol{l}_z(\boldsymbol{k})$ are the *n*-, *m*- and *p*-dimentional measurement or pseudo-measurement vectors, respectively. Usually, one has $\boldsymbol{w}_0(\boldsymbol{k}) = \boldsymbol{o}$.

By applying the principle of least squares to $\boldsymbol{l}_x(\boldsymbol{k})$, $\boldsymbol{l}_w(\boldsymbol{k})$ and $\boldsymbol{l}_z(\boldsymbol{k})$, the identical solution can be obtained as in table 1 [Wang, 1997, 2009]. Moreover, this alternate prospect directly makes the measurement residual vectors available at each epoch for further error analysis in Kalman filter:

$$\boldsymbol{v}_{l_x l_x}(\boldsymbol{k}) = \boldsymbol{D}_{l_x l_x}(\boldsymbol{k})\boldsymbol{D}_{xx}^{-1}(\boldsymbol{k}/\boldsymbol{k}-1)\boldsymbol{K}(\boldsymbol{k})\boldsymbol{d}(\boldsymbol{k}) \tag{13}$$

$$\boldsymbol{v}_{l_w l_w}(\boldsymbol{k}) = \boldsymbol{Q}(\boldsymbol{k}-1)\boldsymbol{B}^T(\boldsymbol{k}-1)\boldsymbol{D}_{xx}^{-1}(\boldsymbol{k}/\boldsymbol{k}-1)\boldsymbol{K}(\boldsymbol{k})\boldsymbol{d}(\boldsymbol{k}) \tag{14}$$

$$\boldsymbol{v}_{l_z l_z}(\boldsymbol{k}) = \{\boldsymbol{C}(\boldsymbol{k})\boldsymbol{K}(\boldsymbol{k}) - \boldsymbol{E}\}\boldsymbol{d}(\boldsymbol{k}) \tag{15}$$

with their variance matrices

$$\boldsymbol{D}_{v_{l_x l_x}}(\boldsymbol{k}) = \boldsymbol{A}(\boldsymbol{k}-1)\boldsymbol{D}_{xx}(\boldsymbol{k}-1)\boldsymbol{A}^T(\boldsymbol{k}-1)\boldsymbol{C}^T(\boldsymbol{k}) \rightarrow \boldsymbol{D}_{dd}^{-1}(\boldsymbol{k})\boldsymbol{C}(\boldsymbol{k})\boldsymbol{A}(\boldsymbol{k}-1)\boldsymbol{D}_{xx}(\boldsymbol{k})\boldsymbol{A}^T(\boldsymbol{k}-1) \tag{16}$$

$$\boldsymbol{D}_{v_{l_w l_w}}(\boldsymbol{k}) = \boldsymbol{Q}(\boldsymbol{k}-1)\boldsymbol{B}^T(\boldsymbol{k}-1)\boldsymbol{C}^T(\boldsymbol{k})\boldsymbol{D}_{dd}^{-1}(\boldsymbol{k}) \rightarrow \boldsymbol{C}(\boldsymbol{k})\boldsymbol{B}(\boldsymbol{k}-1)\boldsymbol{Q}(\boldsymbol{k}-1) \tag{17}$$

$$\boldsymbol{D}_{v_{l_z l_z}}(\boldsymbol{k}) = \{\boldsymbol{E} - \boldsymbol{C}(\boldsymbol{k})\boldsymbol{K}(\boldsymbol{k})\}\boldsymbol{R}(\boldsymbol{k}) \tag{18}$$

Obviously, all of the three residual vectors are the functions of the system innovation vector $\boldsymbol{d}(\boldsymbol{k})$. The measurement residuals are characterized as an uncorrelated series epochwise

$$\boldsymbol{Cov}\{\boldsymbol{v}(\boldsymbol{i}), \boldsymbol{v}(\boldsymbol{j})\} = \boldsymbol{O}\ (\boldsymbol{i} \neq \boldsymbol{j}) \tag{19}$$

as how the system innovation series are characterized.

### 3.2. THE REDUNDANCY CONTRIBUTION IN KALMAN FILTERING

According to the reliability theory in least squares method, one can judge of the redundancy contribution of the individual measurements through the matrix $\boldsymbol{D}_{vv}(\boldsymbol{k})\boldsymbol{P}(\boldsymbol{k})$ or $\boldsymbol{D}_{vv}(\boldsymbol{k})\boldsymbol{D}^{-1}(\boldsymbol{k})$, wherein $\boldsymbol{P}(\boldsymbol{k})$ is the corresponding weight matrix of the measurement vector. In case that the measurements are not correlated to each other, the detailed analysis can be found in (Wang, 1997, 2009).

For three independent measurement groups as from (3) to (5), the individual group redundant indices are generally equal to

$$\mathrm{r}_x(\boldsymbol{k}) = \boldsymbol{tr}[\boldsymbol{A}(\boldsymbol{k}-1)\boldsymbol{D}_{xx}(\boldsymbol{k})\boldsymbol{A}^T(\boldsymbol{k}-1)\boldsymbol{C}^T(\boldsymbol{k})\boldsymbol{D}_{dd}^{-1}(\boldsymbol{k})\boldsymbol{C}(\boldsymbol{k})] \tag{20}$$

$$\mathrm{r}_w(\boldsymbol{k}) = \boldsymbol{tr}[\boldsymbol{Q}(\boldsymbol{k})\boldsymbol{B}^T(\boldsymbol{k})\boldsymbol{C}^T(\boldsymbol{k})\boldsymbol{D}_{dd}^{-1}(\boldsymbol{k})\boldsymbol{C}(\boldsymbol{k})\boldsymbol{B}(\boldsymbol{k}-1)] \tag{21}$$

$$\mathrm{r}_z(\boldsymbol{k}) = \boldsymbol{tr}[\boldsymbol{I} - \boldsymbol{C}(\boldsymbol{k})\boldsymbol{K}(\boldsymbol{k})] \tag{22}$$

wherein "tr" stands for trace. The number of the total redundancy at *k* is given by

$$\mathrm{r}(\boldsymbol{k}) = \mathrm{r}_x(\boldsymbol{k}) + \mathrm{r}_w(\boldsymbol{k}) + \mathrm{r}_z(\boldsymbol{k}) = \mathrm{p}(\boldsymbol{k}) \tag{23}$$

which is exactly equal to the number of the measurements in $\boldsymbol{z(k)}$

### 3.3. VARIANCE-COVARIANCE ESTIMATION

The most widely known method for the varaince and covariance component estimation (VCCE) is the Helmert's method. More on development to simplify, unify or extend the various VCCE algorithms can be found in (Förstner, 1979; Koch, 1986; Ou, 1989; Yu, 1996). The research in this paper adapts the derivation of VCCE algorithm from (Qu, 1989).

Let $\boldsymbol{l}$ be a $n$-dimensional observation vector as the function of the $u$-dimensional parameter vector $\boldsymbol{y}$

$$\boldsymbol{l} = \boldsymbol{F}\boldsymbol{y} + \boldsymbol{\varepsilon} \tag{24}$$

wherein $\boldsymbol{\varepsilon}$ is the observation error vector assumed to be Gaissian white noise with the observation variance matrix

$$\boldsymbol{\Sigma}_{\boldsymbol{ll}} = \begin{pmatrix} \sigma_{11}^2 & \cdots & \sigma_{1i} & \cdots & \sigma_{1n} \\ \vdots & & \vdots & & \vdots \\ \sigma_{i1} & \cdots & \sigma_{\boldsymbol{ii}}^2 & \cdots & \sigma_{in} \\ \vdots & & \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{ni} & \cdots & \sigma_{\boldsymbol{nn}}^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{(0)11}^2 q_{11} & \cdots & \sigma_{(0)1\boldsymbol{i}} q_{1\boldsymbol{i}} & \cdots & \sigma_{(0)1\boldsymbol{n}} q_{1\boldsymbol{n}} \\ \vdots & & \vdots & & \vdots \\ \sigma_{(0)\boldsymbol{i}1} q_{\boldsymbol{i}1} & \cdots & \sigma_{(0)\boldsymbol{ii}}^2 q_{\boldsymbol{ii}} & \cdots & \sigma_{(0)1\boldsymbol{n}} q_{\boldsymbol{in}} \\ \vdots & & \vdots & & \vdots \\ \sigma_{(0)\boldsymbol{n}1} q_{\boldsymbol{n}1} & \cdots & \sigma_{(0)\boldsymbol{ni}} q_{\boldsymbol{ni}} & \cdots & \sigma_{(0)\boldsymbol{nn}}^2 q_{\boldsymbol{nn}} \end{pmatrix} \tag{25}$$

wherein $\sigma_{(0)\boldsymbol{ii}}^2$, $\sigma_{(0)\boldsymbol{jk}}$, $q_{\boldsymbol{ii}}$ and $q_{\boldsymbol{jk}}$ stand for the corresponding variance or covariance of unit weight and the cofactors, respectively. In general, it maintains $\sigma_{(0)\boldsymbol{jk}} q_{\boldsymbol{jk}} = \sigma_{(0)\boldsymbol{kj}} q_{\boldsymbol{kj}}$. In practice, one seeks to estimate either $\sigma_{\boldsymbol{ii}}^2$ and $\sigma_{jk}$ or $\sigma_{(0)\boldsymbol{ii}}^2$ and $\sigma_{(0)\boldsymbol{jk}}$ by taking the advantages of the available measurement residuals.

The least sqaures solution delivers the residual vector $\boldsymbol{v}$ of $\boldsymbol{l}$ with its variance matrix $\boldsymbol{D}_{\boldsymbol{vv}}$. Statistically, the following equation exists for the expectation of weighted sum of the residuals squared

$$\boldsymbol{E}(\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{P}\boldsymbol{v}) = \boldsymbol{tr}((\boldsymbol{Q}_{\boldsymbol{vv}}\boldsymbol{P})^{\boldsymbol{T}}\boldsymbol{P}\boldsymbol{\Sigma}_{\boldsymbol{ll}}) \tag{26}$$

whereas $\boldsymbol{v}$ and $\boldsymbol{P}$ are the residual vector and weight matrix of $\boldsymbol{l}$, respectively, and $\boldsymbol{Q}_{\boldsymbol{vv}}$ is the cofactor matrix of $\boldsymbol{v}$. Through appropriate partition, the following normal equation for the individual variance-covariance factors is given by (Ou, 1989; etc.):

$$\boldsymbol{w} = \boldsymbol{M}\boldsymbol{\theta} \tag{27}$$

with

$$\boldsymbol{\theta} = \left(\sigma_{(0)11}^2 \quad \sigma_{(0)12} \quad \cdots \quad \sigma_{(0)1\boldsymbol{n}} \quad \sigma_{(0)22}^2 \quad \cdots \quad \sigma_{(0)2\boldsymbol{n}} \right.$$

$$\left. \cdots \quad \sigma_{(0)(\boldsymbol{n}-1)(\boldsymbol{n}-1)}^2 \quad \sigma_{(0)(\boldsymbol{n}-1)\boldsymbol{n}} \quad \sigma_{(0)\boldsymbol{n},\boldsymbol{n}}^2 \right)^{\boldsymbol{T}} \tag{28}$$

$$\boldsymbol{w} = \left(p_{11}v_{11}^2 \quad 2p_{12}v_1v_2 \quad \cdots \quad 2p_{1\boldsymbol{n}}v_1v_{\boldsymbol{n}} \quad p_{22}v_{22}^2 \quad \cdots \right.$$

$$2p_{2\boldsymbol{n}}v_2v_{\boldsymbol{n}} \quad \cdots \quad p_{(\boldsymbol{n}-1)(\boldsymbol{n}-1)}v_{\boldsymbol{n}-1}^2 \quad 2p_{(\boldsymbol{n}-1)\boldsymbol{n}}v_{\text{n-1}}v_{\boldsymbol{n}}$$

$$\left. p_{\boldsymbol{nn}}v_{\boldsymbol{n}}^2 \right)^{\boldsymbol{T}} \tag{29}$$

$$\boldsymbol{M}_{\boldsymbol{jk}} = \boldsymbol{tr}((\boldsymbol{Q}_{\boldsymbol{vv}}\boldsymbol{P})^{\boldsymbol{T}}\boldsymbol{P}_{\boldsymbol{j}}\boldsymbol{T}_{\boldsymbol{k}}) \tag{30}$$

wherein $p_{\boldsymbol{ij}}$ is the elements intersected between the $\boldsymbol{i}$th row and the $\boldsymbol{j}$th column in $\boldsymbol{P} = \Sigma_{\boldsymbol{ll}}^{-1}$, and

$$\boldsymbol{P}_{\boldsymbol{j}} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & p_{\boldsymbol{jj}} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \text{ or } = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & p_{\boldsymbol{jk}} & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & p_{\boldsymbol{kj}} & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \tag{31}$$

$$\boldsymbol{T}_{\boldsymbol{k}} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & q_{\boldsymbol{kk}} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \text{ or } = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & q_{\boldsymbol{kj}} & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & q_{\boldsymbol{jk}} & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \tag{32}$$

If multiple variance or covariance factors are assumed to be equal, $\boldsymbol{P}_{\boldsymbol{j}}$ and $\boldsymbol{T}_{\boldsymbol{k}}$ will be changed accordingly so that it may have more non-zero elements in them, and the dimension of $\boldsymbol{\theta}$ may be reduced as well.

Furthermore, the matrix $\boldsymbol{Q}_{\boldsymbol{vv}}\boldsymbol{P}$ has the total redundancy number $r_{\boldsymbol{l}}$ that $\boldsymbol{l}$ possesses. At most $r_{\boldsymbol{l}}(r_{\boldsymbol{l}}+1)/2$ independent variance and covariance components can be uniquely determined [Xu et al, 2007].

In general, the non-diagonal elements of the coefficient matrix $\boldsymbol{M}$ in (24) are much smaller than its diagonal ones so that the successive approximation has practically been made to ignore the non-diagonal elements of $\boldsymbol{M}$ (Förstner, 1979; Ou, 1989). Therefore, the following

solution of the variance and covariance factors is given as follows

$$\boldsymbol{\sigma}_{ii}^2 = \mathrm{p}_{ii}\mathrm{v}_i^2 / \boldsymbol{M}_{ii} \tag{33}$$

$$\boldsymbol{\sigma}_{jk} = 2\mathrm{p}_{jk}\mathrm{v}_\mathrm{j}\mathrm{v}_\mathrm{k} / \mathrm{r}_\mathrm{jk} \tag{34}$$

with

$$\mathrm{r}_\mathrm{jk} = \boldsymbol{tr}((\boldsymbol{Q}_{vv}\boldsymbol{P})^T \boldsymbol{P}_j\boldsymbol{T}_k) + \boldsymbol{tr}((\boldsymbol{Q}_{vv}\boldsymbol{P})^T \boldsymbol{P}_k\boldsymbol{T}_j)) \tag{35}$$

The result given above in this subsection is only corresponding to the variance and covariance components estimation at a single epoch, namely $\boldsymbol{k}$ here. In Kalman filtering, the residuals from previous multiple epochs, even all of the past epochs can be used to accumulatively estimate the variance and covariance components of $\boldsymbol{R}(\boldsymbol{k})$ and $\boldsymbol{Q}(\boldsymbol{k})$ so that the estimation quality can be improved. Accordingly, (33) and (34) become

$$\boldsymbol{\sigma}_{ii}^2(\boldsymbol{k}) = \frac{\sum_{k-k_0}^{k} \mathrm{p}_{ii}(\boldsymbol{k})\mathrm{v}_i^2(\boldsymbol{k})}{\sum_{k-k_0}^{k} \boldsymbol{M}_{ii}(\boldsymbol{k})} \tag{36}$$

$$\boldsymbol{\sigma}_{jk}(\boldsymbol{k}) = \frac{2\sum_{k-k_0}^{k} \mathrm{p}_{jk}(\boldsymbol{k})\mathrm{v}_\mathrm{j}(\boldsymbol{k})\mathrm{v}_\mathrm{k}(\boldsymbol{k})}{\sum_{k-k_0}^{k} \mathrm{r}_\mathrm{jk}(\boldsymbol{k})} \tag{37}$$

where $k_0$ is the number of the used previous epochs. In practice, the equations (33), (34) and (36), (37) can further be reformulated to fit to other reasonable assumptions, for example, the measurement variances should be identical in GPS positioning if the satellites have the same elevation angles or if their elevation differences are within a given range (e.g. 5 degrees).

By the way, readers are referred to (Wang, 1997; Wang et al, 2009) for the case that $\boldsymbol{R}(\boldsymbol{k})$ and $\boldsymbol{Q}(\boldsymbol{k})$ are diagonal, i.e. the measurements and process noise factors are uncorrelated.

### 3.4 Kinematic Relative GPS Positioning

The objective of kinematic relative position is to determine the coordinates of moving object $\boldsymbol{A}$ with respect to a stationary known point $\boldsymbol{B}$. This type of positioning requires simultaneous observations at two points and can be performed with code ranges or/and carrier phases (Hofmann-Wellenhof et al., 2008).

The system equation is given as follows

$$\begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \\ \Delta\nabla\boldsymbol{N} \end{bmatrix}_k = \begin{bmatrix} 1 & \Delta\boldsymbol{t}_k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \\ \Delta\nabla\boldsymbol{N} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{2}\Delta\boldsymbol{t}_k^2 \\ \Delta\boldsymbol{t} \\ 0 \end{bmatrix}_{k-1} [\ddot{\boldsymbol{x}}]_{k-1} \tag{38}$$

where $\boldsymbol{x}$ is the position vector, $\dot{\boldsymbol{x}}$ is the velocity vector, $\ddot{\boldsymbol{x}}$ is the acceleration vector all in ECEF, $\nabla\Delta\boldsymbol{N}$ is the double differenced ambiguity vector and $\Delta\boldsymbol{t}_k = \boldsymbol{t}_k - \boldsymbol{t}_{k-1}$. $\ddot{\boldsymbol{x}}$ is modeled as process noise.

$$\ddot{\boldsymbol{x}} \sim \boldsymbol{N}\left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{\ddot{X}}^2 & 0 & 0 \\ 0 & \sigma_{\ddot{Y}}^2 & 0 \\ 0 & 0 & \sigma_{\ddot{Z}}^2 \end{bmatrix} \right) \tag{39}$$

Double-differenced measurements are usually used and the measurement equation for a single frequency receiver is given by:

$$\Delta\nabla\boldsymbol{P}_{\boldsymbol{AB}}^{\boldsymbol{jk}} = \Delta\nabla\rho_{\boldsymbol{AB}}^{\boldsymbol{jk}} + \Delta\nabla\varepsilon_{\boldsymbol{P,AB}}^{\boldsymbol{jk}} \tag{40}$$

$$\lambda_1\Delta\nabla\phi_{\boldsymbol{AB}}^{\boldsymbol{jk}} = \Delta\nabla\rho_{\boldsymbol{AB}}^{\boldsymbol{jk}} + \lambda_1\Delta\nabla\boldsymbol{N}_1 + \Delta\nabla\varepsilon_{\boldsymbol{L1,AB}}^{\boldsymbol{jk}} \tag{41}$$

where $\Delta\nabla\boldsymbol{P}_{\boldsymbol{AB}}^{\boldsymbol{jk}}$ is the double differenced L1 code range measurement between satellite $\boldsymbol{j}$ and satellite $\boldsymbol{k}$ and receiver $\boldsymbol{A}$ and receiver $\boldsymbol{B}$ in meters, $\Delta\nabla\phi_{\boldsymbol{AB}}^{\boldsymbol{jk}}$ is the corresponding double differenced carrier phase measurement in cycles, $\Delta\nabla\rho_{\boldsymbol{AB}}^{\boldsymbol{jk}}$ is the geometric range, $\lambda_1$ is the wave length of L1 carrier phase, $\nabla\Delta\boldsymbol{N}_1$ is the double differenced ambiguity unknown of L1 carrier phase, $\Delta\nabla\varepsilon_{\boldsymbol{P,AB}}^{\boldsymbol{jk}}$ is the code measurement noise and $\Delta\nabla\varepsilon_{\boldsymbol{L1,AB}}^{\boldsymbol{jk}}$ is the phase measurement noise, respectively. For short baselines, the residual tropospheric and ionospheric effects in double differenced measurements are assumed to be negligibly small and are ignored. The double difference linear combination in (29) and (30) introduces measurement correlations and the corresponding variance-covariance matrix is as follows:

$$\boldsymbol{R}_{\Delta\nabla} = \begin{pmatrix} \sigma_{\boldsymbol{A},j}^2 + \sigma_{\boldsymbol{B},j}^2 + \sigma_{\boldsymbol{A},1}^2 + \sigma_{\boldsymbol{B},1}^2 & \sigma_{\boldsymbol{A},j}^2 + \sigma_{\boldsymbol{B},j}^2 \\ \sigma_{\boldsymbol{A},j}^2 + \sigma_{\boldsymbol{B},j}^2 & \sigma_{\boldsymbol{A},j}^2 + \sigma_{\boldsymbol{B},j}^2 + \sigma_{\boldsymbol{A},2}^2 + \sigma_{\boldsymbol{B},2}^2 \\ \vdots & \vdots \\ \sigma_{\boldsymbol{A},j}^2 + \sigma_{\boldsymbol{B},j}^2 & \sigma_{\boldsymbol{A},j}^2 + \sigma_{\boldsymbol{B},j}^2 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \sigma_{A,j}^2 + \sigma_{B,j}^2 \\ \cdots & \sigma_{A,j}^2 + \sigma_{B,j}^2 \\ \vdots & \vdots \\ \cdots & \sigma_{A,j}^2 + \sigma_{B,j}^2 + \sigma_{A,n}^2 + \sigma_{B,n}^2 \end{pmatrix} \quad (42)$$

where $\sigma_{A,j}^2$ and $\sigma_{B,j}^2$ are the variances for the measurements to the reference satellite $j$ while $\sigma_{A,k}^2$ and $\sigma_{B,k}^2$ for $k = 1,...,n$ are the variances for the measurements to the remaining satellites. The algorithm can only estimate the variance and covariance factors for each entry in the matrix. Therefore, the individual variances cannot be separated. Thus, (42) is written as

$$\boldsymbol{R}_{\Delta\nabla} = \begin{pmatrix} \sigma_1^2 & \sigma_j^2 & \cdots & \sigma_j^2 \\ \sigma_j^2 & \sigma_2^2 & \cdots & \sigma_j^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_j^2 & \sigma_j^2 & \cdots & \sigma_n^2 \end{pmatrix} \quad (43)$$

For each epoch there are a maximum of $n+1$ variances factors that can be estimated.

The elevation angle of the satellite influences the signal path length and the strength of the received signal (Hofmann-Wellenhof et al., 2008). As the elevation of the satellites decreases, the signal-to-noise ratio increases. In the VC estimation algorithm the measurements can accordingly be grouped based on the elevation angles.

## 4. TEST AND RESULTS

The algorithm described in the previous section was implemented in GPS RTK positioning. Simulated and real static GPS data were used and the processing results are represented and discussed in this section. For demonstration purposes only L1 pseudorange and carrier phase measurements were used. Furthermore the 3D acceleration of the vehicle is modeled as the process noise. The integer ambiguities were resolved using the LAMBDA method (Teunissen et al, 1997). The initial variance assigned to the process noise, code and phase measurements were arbitrary and independent of satellite elevation angles. The 3D acceleration process noise components are estimated. The floated ambiguity process noise is assumed to be zero. The variance-covariance components were grouped against satellite elevation and it is assumed that for short intervals in satellite elevation angles the variance does not change much. The ranges of the satellite elevation angles for each group reasonably vary so that each group can have almost the same number of measurements. This ensures that there are enough residuals to estimate all of the variance components reliably. As a result there will be uneven intervals and also be different for each dataset.

### 4.1 RESULTS FROM SIMULATED GPS DATA

The undifferenced code and phase kinematic measurements were simulated at 1Hz for 4800 seconds with an elevation mask of 10 degrees. The simulation algorithm is as follows:

The ECEF acceleration process noise variance is set at a constant value of

$$\begin{bmatrix} \boldsymbol{\sigma}_{\ddot{X}}^2 & \boldsymbol{\sigma}_{\ddot{Y}}^2 & \boldsymbol{\sigma}_{\ddot{Z}}^2 \end{bmatrix}^T = \begin{bmatrix} 0.10^2 & 0.15^2 & 0.20^2 \end{bmatrix}^T [\mathrm{m}^2/\mathrm{s}^4] \quad (44)$$

The measurement noise variance applied to the measurements varies with the satellite elevation angle based on

$$\sigma_E^2 = \sigma_{90}^2 \left[ 0.5 + 0.5 \exp(17.5/E) \right] \quad (45)$$

where $\sigma_{90}^2$ is the variance at the zenith and $E$ is the satellite elevation angle in degrees. The zenith variances used for code and phase measurements are $(0.300\boldsymbol{m})^2$ and $(0.003\boldsymbol{m})^2$, respectively. In this test the initial Q for the process noise vector is diag[$0.35^2$ $0.35^2$ $0.35^2$][$\boldsymbol{m}^2$]. The initial double differenced code and phase variances are $(1.200\boldsymbol{m})^2$ and $(0.012\boldsymbol{m})^2$.

Fig. 1 shows the process noise standard deviation estimation as a function of time with its true value. The results show that the process noise can be estimated with 400 epochs or less.

Fig. 2 shows the covariance estimation as a function of time together with its true value. The results show that the covariance component for the code and phase measurement can be estimated with at least 250 and 200 epochs respectively.

Fig. 3 and 4 show the standard deviation (SD) estimation of the code and phase measurements as a function of time together with its true value. Three groups are presented, one with the highest, average and lowest elevation angles. The estimation of the first group (Elevation 65 to 72) starts at 3806 seconds because it the time when one or more satellites enter this elevation range. Similarly the third group starts at 583 seconds. The results show that the algorithm takes at least 120 epochs and 171 to estimate the code and phase standard deviations.

Fig. 5 shows the estimated standard deviation for all the groups as a function of satellite elevation angles together with its corresponding initial and true value plots. The estimated components fit closely to the true curve.

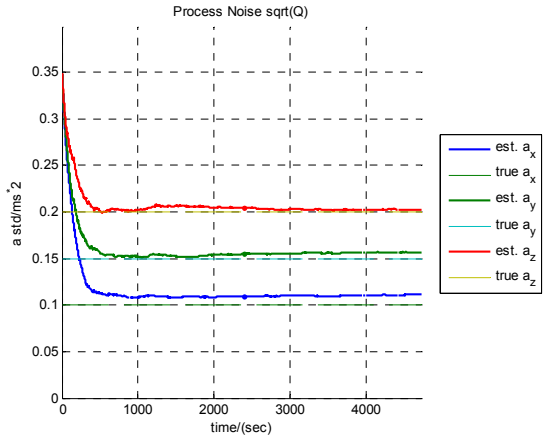


Fig. 1 Process noise vs. time

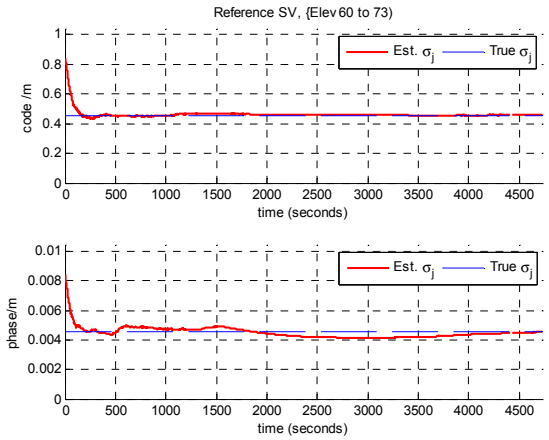


Fig. 2 Covariance vs. time

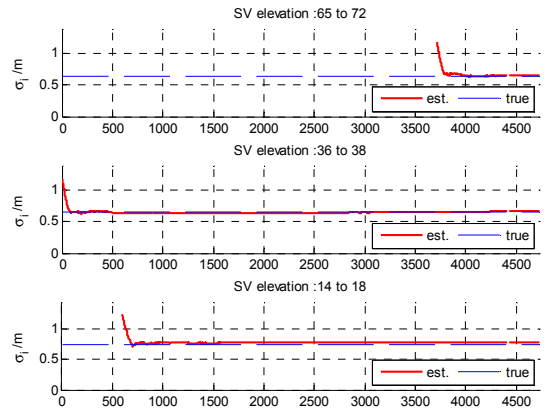


Fig. 3 Selected code SD vs. time

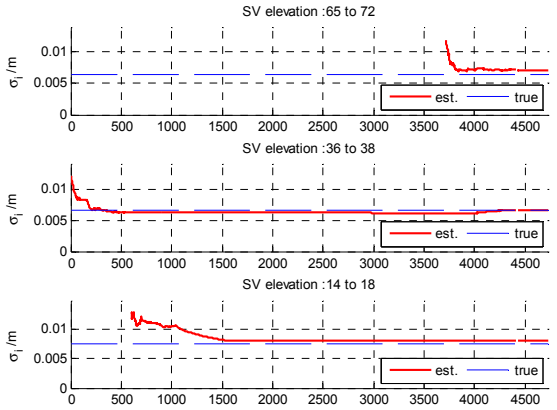


Fig. 4 Selected phase SD vs. time

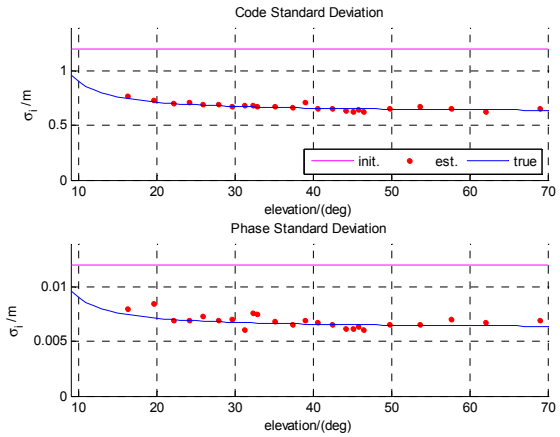


Fig. 5 Code and phase SD vs. Satellite Elevation

Fig. 6 and 7 show the position error together the estimated standard deviation for two cases; first without AKF and second with AKF.

The estimated standard deviation is more realistic with AKF. Fig. 8 and 9 show the probability density functions (PDFs) of standardized position error without and with the application of AKF together with a standardized normal distribution curve. The PDFs with the AKF algorithm fits the standardized normal distribution curve which suggests that the estimated variances for the states represent the true distribution of the states.

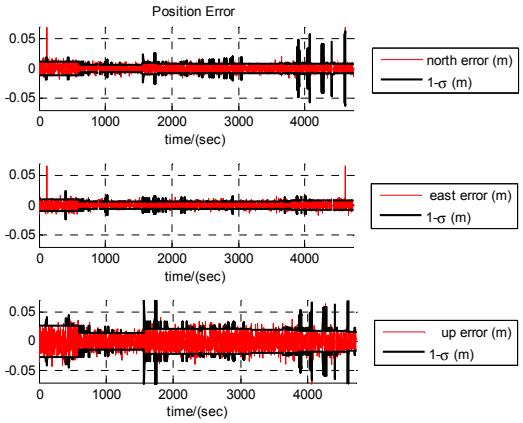


Fig. 6 Position error without AKF

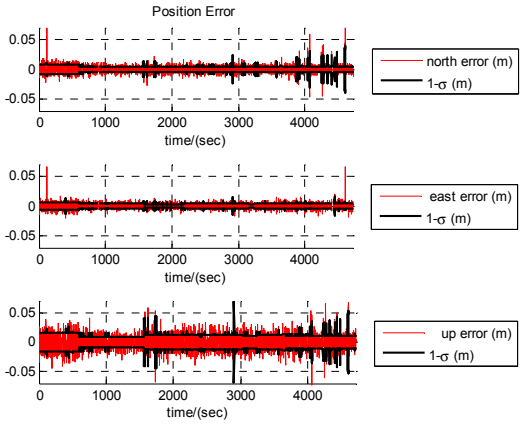


Fig. 7 Position error with AKF

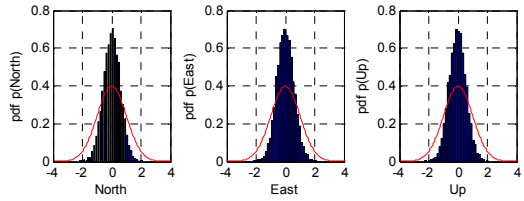


Figure 8 PDFs of standardized position error using initial variances (i.e. without AKF)

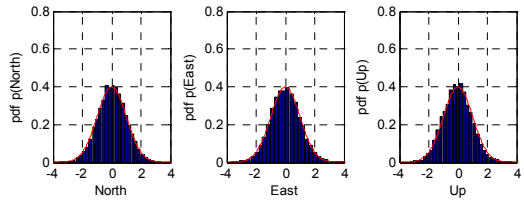


Fig. 9 PDFs of standardized position error with AKF

### 4.2 RESULTS FROM REAL GPS DATA

This section presents the processing results using the VC estimation algorithm on real GPS data. The data was collected on 24 April 2010 using two Leica 1200 receivers for the duration of 80 minutes. The base station was located at N43°46'26.34512", W79°30'43.23784", 158.697[m] on the Keele Campus of York University, Toronto, Ontario, Canada. The rover trajectory is shown in fig. 10 with its velocity profile as in fig. 11. Both receivers were set at sampling rate of 1.0 Hz and with an elevation mask of 10 degrees. The maximum baseline length and velocity are 3.5 km and 23 m/s respectively. POSGNSS (Waypoint® Software) was used to generate a reference solution (the 'true' rover position).

In this test the initial Q for the process noise vector is diag[$0.75^2$ $0.75^2$ $0.75^2$][$m^2/s^4$]. The initial double differenced code and phase variances are $(0.800\boldsymbol{m})^2$ and $(0.016\boldsymbol{m})^2$.

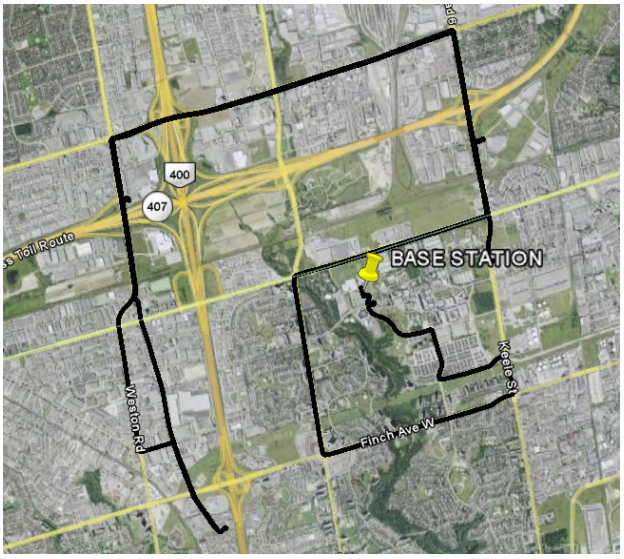


Fig. 10 Kinematic dataset trajectory

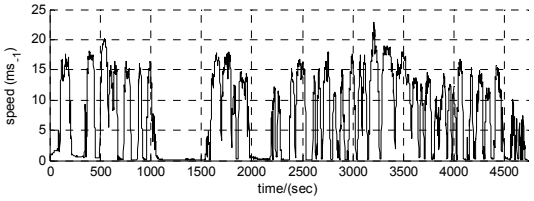


Fig. 11 Speed vs. time

Fig. 12 shows the estimated standard deviations of the process noise factors as a function of time. Fig. 13 shows the covariance estimation as a function of time. The estimated covariance component for code measurement

stabilizes after 210 epochs while the one for the phase measurement takes significantly longer to be convergent.

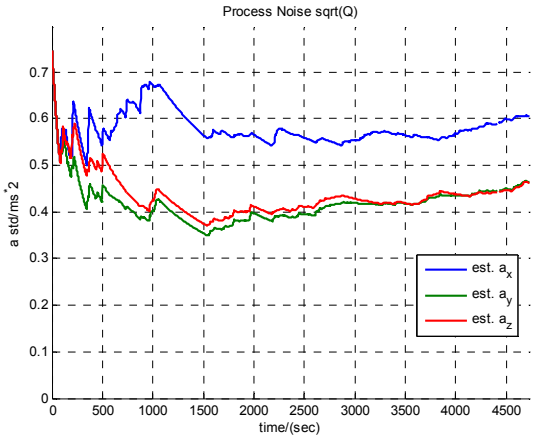


Fig. 12 Process noise vs. time

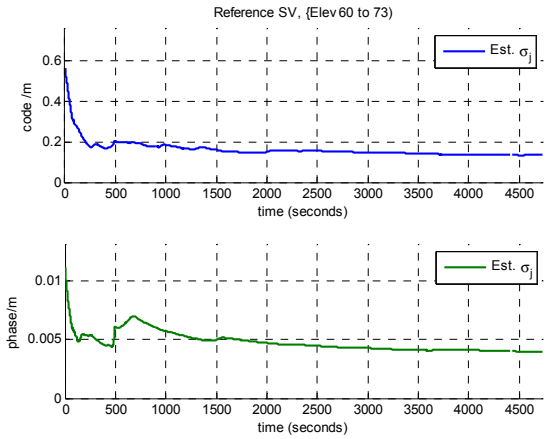


Fig. 13 Covariance vs. time

Fig. 14 and 15 show the estimated standard deviation of the code and phase measurements as a function of time together. Three measurement groups are presented, one with the highest, average and lowest elevation angles.

Fig. 16 shows the estimated standard deviation for all the groups as a function of satellite elevation angles together with its initial standard deviations. Fig. 17 and 18 show the position errors together with the corresponding standard deviation before and after the application of the AKF algorithm. The reference solution contains errors as it comes from POSGNSS commercial software.

Fig. 19 and 20 show the probability density functions (PDFs) of standardized position error without and with the application of AKF together with a standardized normal distribution curve.

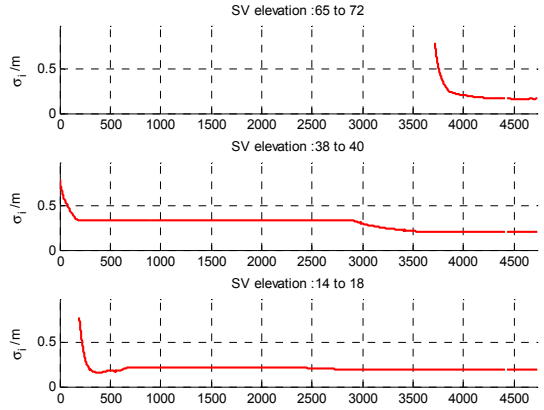


Fig. 14 Selected code SD vs. time

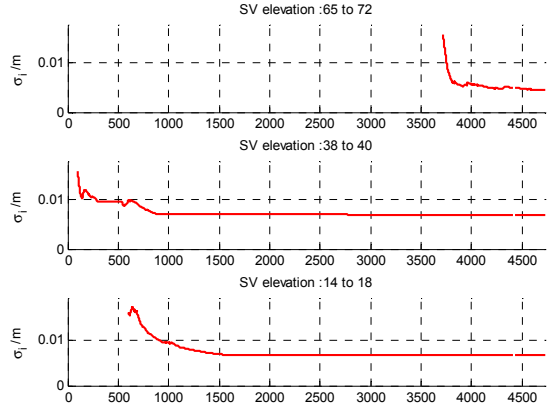


Fig. 15 Selected phase SD vs. time

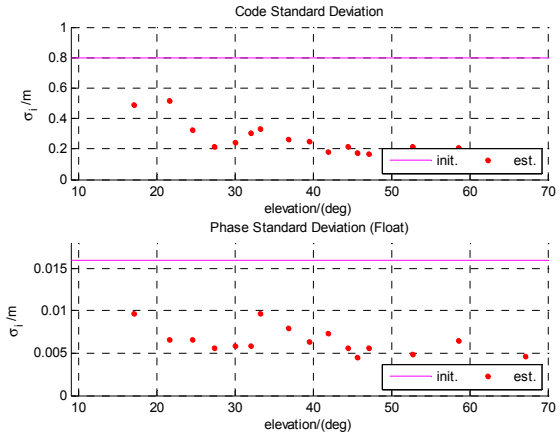


Fig. 16 Code and phase SD vs. Satellite Elevation

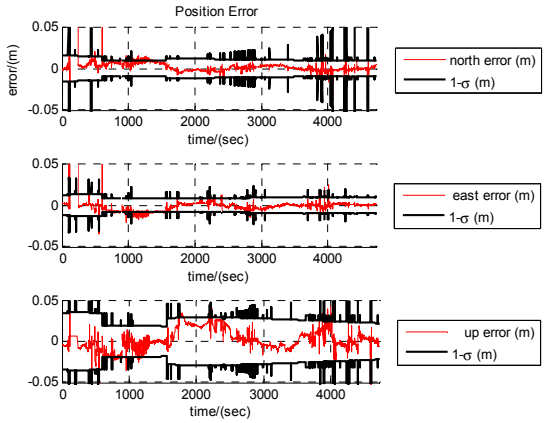


Fig. 17 Position error with without AKF

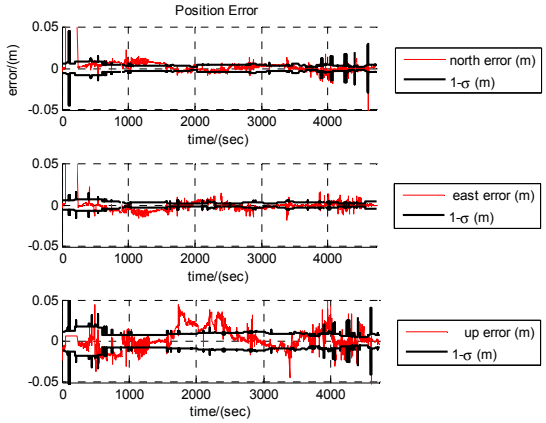


Fig. 18 Position error with with AKF

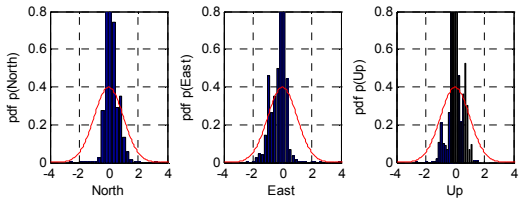


Figure 19 PDFs of standardized position error using initial variances (i.e. without AKF)

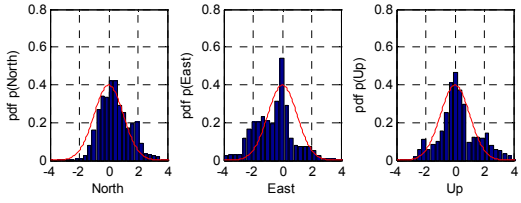


Fig. 20 PDFs of standardized position error with AKF

## 5. REMARKS AND CONCLUSIONS

The paper presents an adaptive Kalman filtering algorithm based on variance and covariance component estimation. The distinction of the proposed AKF comparing with others in the literature lies in its capability of the simultaneous estimation of the variance and covariance factors for both of the measurement vector and the process noise vector. The results from simulated and real datasets showed the feasibility, efficiency and practicality of the algorithm. Furthermore the estimated variance and covariance factors possess good convergence. The authors will further apply this AKF to the integrated navigation and study how the system quality control can benefit from it.

## 6. ACKNOWLEDGEMENTS

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## 7. REFERENCE:

Caspary, Wilhelm; Wang, Jianguo (1998): ”***Redundanzanteile und Varianz-komponenten im Kalman Filter***”**,** No. 4, 1998, ***ZfV*** (Zeitschrift für Vermessung).

Ding, Weidong; et al (2007): ***Improving Adaptive Kalman Estimation in GPS/INS Integration***, Journal of Navigation, vol. 60, pp. 517-529.

Förstner, W. (1979): ***Ein Verfahren zur Schätzunug von Varianz- und Kovarianz-komponenten***, AVN, No. 11-12, 1979, pp. 446-453.

Hu, Congwei; et al (2003): ***Adaptive Kalman filtering for Vehicle Navigation***, Journal of GPS, vol. 2, No. 1, pp. 42-47.

Hu, Congwei and Liu, Dajie (2002): 基于方差分量估计原理的自适应卡尔曼滤波及其应用 (***Adaptive Kalman Filtering and Its Application Based on Variance Component Estimation***), Journal of Zhengzhou Institute of Surveying and Mapping, China, Vol. 19, No. 1, 2002, pp. 15-18.

Jazwinski, A.H. (1969): ***Adaptive Filtering***, Automatica, Vol. 5, pp. 475-485, 1969.

Koch, K.R. (1986): ***Maximum likelihood estimate of variance components*** - Ideas by A.J. Pope, Bull. Géod. Vol. 60, pp. 329-338, 1986.

Louv, William (1984): ***Adaptive Filtering***, Technometrics, Vol. 26, No. 4, pp. 399-409, 1984.

Mehra, R. K. Seereeram, S., Bayard, D., Hadaegh, F. (1995): ***Adaptive Kalman Filtering, Failure Detection and Identification for Spacecraft Attitude Estimation***, *4th IEEE Conference on Control Applications*, 1995, pp. 176-181.

Mehra, R. K. (1970): ***On the Identification of Variances and Adaptive Kalman Filtering***, IEEE Transactions on Automatic Control, Vol. AC-15, 1970, pp. 175-184.

Mohamed, A.H and Schwarz, K. P. (1999): ***Adaptive Kalman filtering for INS/GPS***, Journal of Geodesy, Vol. 73, pp. 193-203.

Ou, Ziqiang (1989), ***Estimation of Variance and Covariance Components***, Bull, Géod. 63 (1989) pp. 139-148.

Oussalah, M. and Schutter J.De (2000): ***Adaptive Kalman filter for noise identification***, ISMA25, September 2000, pages 1225–1232, 2000. URL: http://www.ismaisaac.be/publications/PMA_MOD _ publications/ISMA25/p1225p1232.pdf.

Teunissen, P.J.G., P.J. de Jonge, and C.C.J.M Tiberius (1997), ***The least-squares ambiguity decorrelation adjustment: its performance on short GPS baselines and short observation spans***, Journal of Geodesy, Vol. 71, No. 10, pp. 589-602.

Wang, Jian-Guo (2009): ***Reliability Analysis in Kalman Filtering***, Journal of GPS, Vol. 8, No. 1, pp.101-111.

Wang, Jian-Guo; Gopaul, Nilesh and Scherzinger, Bruno (2009): ***Simplified Algorithms of Variance component Estimation for Static and Kinematic GPS Single Point Positioning***, Journal of GPS, Vol. 8, No. 1, pp.43-52.

Wang, Jian-Guo (1997): ***Filtermethoden zur fehlertoleranten kinematischen Positions-bestimmung***, No. 52, ***Schriftenreihe Studiengang Vermessungswesen der Universität der Bundeswehr München***, Neubiberg, Germany 1997.

Xu, Peiliang; Liu, yumei; Shen, Yunzhong (2007), ***Estimability analysis of variance and covariance components***, Journal of Geodesy, Vol. 81, No.9, 2007, pp. 593-602.

Yang, Yuanxi; Gao, Weiguang (2006): ***An optimal adaptive Kalman filter***, Journal of Geodesy, vol. 80, No. 4, pp. 177-183.

Yang, Yuanxi; Xu, Tianhe (2003): ***An adaptive Kalman filter based on Sage Windowing Weights and Variance components***, Journal of Navigation, vol. 56, pp. 231-240.

Yang, Y.; Hu, H. and Xu, G. (2001): ***Adaptively robust filtering for kinematic geodetic positioning,*** Journal of Geodesy, vol. 75, pp. 109-116.

Yu, Z.C. (1996): ***A universal formula of maximum likelihood estimation of variance-covariance components***, Journal of Geodesy, vol. 70, pp. 233-240.

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