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On Posteriori Variance and Covariance Components Estimation in GPS Relative Positioning

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ABSTRACT

This paper focuses on realization of the variance and covariance component estimation in static relative GPS positioning for the double-differenced measurements in sequential least squares. The algorithm presented is based on the algorithm from [Ou, 1989] through the incorporation into the sequential least-squares (Wang et al, 2009; etc.). It is practical and easy to implement, and computationally efficient. Numerical results from simulated and real static GPS data in relative positioning mode are presented and discussed.

Key words: variance-covariance component estimation, sequential least squares, static GPS, relative positioning, redundancy contribution.

1. INTRODUCTION

The least-squares (LS) method requires an accurate stochastic model of the measurements in order to produce realistic estimates of the unknowns. In general, the a priori measurement variance-covariance (VC) matrix can only be determined based on the limited available knowledge. Sometimes, it can be partially or completely unknown. The classical method for determining the VC matrix is through empirical analysis on measurement errors or manually in an ad hoc fashion. The results from empirical analysis may not quantitatively represent the real stochastic model. As a common practice, one usually seeks to estimate the VC components on the basis of the collected measurements.

The algorithm development and application for the VC component estimation attracted considerable research attentions from time to time. Examples of early studies, since Helmert [1907], include the work in [Förstner, 1979; Grafarend, et al, 1980; Li, 1983; Koch, 1986; Ou, 1989; Yu, 1996; Xu, et al, 2006, 2007; Teunissen and Amiri-Simkooei, 2008; and Amiri-Simkooei, 2007; etc.]. Recently, more and more application-based studies have been introduced [Sieg and Hirsch, 2000; Wang & Rizos, 2002; Tiberius, 2003; Rietdorf, 2004; Tesmer, 2004; Zhou, et al, 2006; Amiri-Simkooei, 2007; Bähr, et al, 2007; Böckmann, 2008; Milbert, 2008; Wang et al, 2009 etc.].

Various VC estimation algorithms have been developed based on different estimation principles and distribution assumptions. The estimator after Helmert is probably the most popular VC estimation algorithm [Förstner, 1979; Koch, 1986; Cui et al, 2001; Bähr, et al, 2007; etc.], which is rigorous and requires considerable amount of computation time as well. Different simplifications and approximations were made for specific practical reasons. A good summary about them can be found in [Hermann et al, 2007; Cui, et al, 2001]. For further information, refer to [Welsch, 1978; Kubik, 1967; Förstner, 1979; Persson, 1980; etc.].

Förstner [1979] estimated the variance components using the measurement redundant contribution together with the measurement residuals. One major drawback of this method is it assumes that the variance matrix is diagonal and therefore cannot be used for correlated measurements. Ou [1989] made an extension to Förstner's method to estimate the variance and covariance factors simultaneously.

This paper focuses on realization of the variance and covariance component estimation for the double-differenced measurements in static relative GPS positioning in the fashion of sequential least squares. The approach is based on the algorithm from [Ou, 1989] through the incorporation into the sequential least-squares (Wang et al, 2009; etc.). The algorithm presented is practical and easy to implement, and computationally efficient. Section 2 first summarizes the algorithm of variance and covariance component estimation and then extends it to the sequential least squares. The last subsection in section 2 describes the real situation for the double differenced measurements in relative point positioning. To illustrate the proposed method, simulated and real static GPS data were processed. The corresponding numerical results and analysis are presented in section 3. In our numerical example the VC matrix of the correlated double-differenced code and carrier phase measurements is estimated. Conclusions and remarks are given in section 4.

2. VARIANCE-COVARIANCE COMPONENTS ESTIMATION

2.1 VC Estimation using Redundancy Index

Let the linearized system of observation equations be given by:

$$\boldsymbol{L}+\boldsymbol{v}=\boldsymbol{B}\delta\hat{\boldsymbol{x}}+\boldsymbol{F}(\boldsymbol{x}^{(0)}) \tag{1}$$

wherein $\boldsymbol{L}$ is the $\boldsymbol{n}\times 1$ observation vector, $\boldsymbol{v}$ is the $\boldsymbol{n}\times 1$ residual vector of $\boldsymbol{L}$, $\boldsymbol{x}$ is the $\boldsymbol{t}\times 1$ parameter vector with its approximate vector as $\boldsymbol{x}^{(0)}$ and its correction vector $\delta\hat{\boldsymbol{x}}$, $\boldsymbol{F}(\boldsymbol{x})$ is the $\boldsymbol{n}\times 1$ vector of non-linear function of x, $\boldsymbol{B}$ is the $\boldsymbol{n}\times\boldsymbol{t}$ design matrix containing the partial derivatives of $\boldsymbol{F}(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ at $\boldsymbol{x}^{(0)}$. The observation vector $\boldsymbol{L}$ is normally distributed with its expectation vector $\tilde{\boldsymbol{L}}$ and its variance-covariance (VC) matrix $\boldsymbol{D}_{LL}$. In practice, the measurement variance-covariance (VC) matrix $\boldsymbol{D}_{LL}$ may be given in different forms as follows:

$$\boldsymbol{D}_{LL}=\boldsymbol{\sigma}_0^2\boldsymbol{P}^{-1}=\boldsymbol{\sigma}_0^2\boldsymbol{Q}=\boldsymbol{\sigma}_0^2\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn}\end{pmatrix} \tag{2}$$

where $q_{ij}=q_{ji}$ for $i\neq j$, $\boldsymbol{\sigma}_0^2$ is the variance of unit weight and $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the weight and cofactor matrix of $\boldsymbol{L}$, respectively.

The least-squares solution delivers

$$\delta\hat{\boldsymbol{x}}=\boldsymbol{N}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{l} \tag{3}$$

with its variance-covariance matrix

$$\boldsymbol{D}_{\hat{x}\hat{x}}=\hat{\boldsymbol{\sigma}}_0^2\boldsymbol{N}^{-1} \tag{4}$$

wherein are

$$\boldsymbol{N}=\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} \tag{5}$$

$$\hat{\boldsymbol{\sigma}}_0^2=\frac{\boldsymbol{v}^T\boldsymbol{P}\boldsymbol{v}}{\boldsymbol{n}-\boldsymbol{t}},(\boldsymbol{n}>\boldsymbol{t}) \tag{6}$$

$$\boldsymbol{l}=\boldsymbol{L}-\boldsymbol{F}(\boldsymbol{x}^{(o)}) \tag{7}$$

The model given above usually assumes to have the fixed relative accuracy relationship between observations because $\boldsymbol{P}$ (or Q) is specified as known. Only the a-priori variance of unit weight $\boldsymbol{\sigma}_0^2$ is estimated using the measurement residuals.

In practice, $\boldsymbol{P}$ is determined using the best available stochastic information about the measurements. However, the choice of $\boldsymbol{P}$ may not be obvious and needs to be improved. To this end, the measurement residuals may be the best available sources to be used.

Consider the variance and covariance component estimation for the individual measurements or practically for independent measurement groups. The VC matrix can be written as

$$\boldsymbol{D}_{LL}=\begin{pmatrix} \sigma_{(0)11}^2 q_{11} & \sigma_{(0)12}q_{12} & \cdots & \sigma_{(0)1n}q_{1n} \\ \sigma_{(0)21}q_{21} & \sigma_{(0)22}^2 q_{22} & \cdots & \sigma_{(0)2n}q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{(0)n1}q_{n1} & \sigma_{(0)n2}q_{n2} & \cdots & \sigma_{(0)nn}^2 q_{nn}\end{pmatrix} \tag{8}$$

wherein $\boldsymbol{\sigma}_{(0)ij}\boldsymbol{q}_{ij}=\boldsymbol{\sigma}_{(0)ji}\boldsymbol{q}_{ji}$ (for $i\neq j$). The variance components $\boldsymbol{\sigma}_{11}^2$, $\boldsymbol{\sigma}_{12}$, $\cdots$, $\boldsymbol{\sigma}_{1n}$, $\boldsymbol{\sigma}_{22}^2$, $\cdots$, $\boldsymbol{\sigma}_{2n}$, $\boldsymbol{\sigma}_{33}^2$, $\cdots$, $\boldsymbol{\sigma}_{nn}^2$ or $\boldsymbol{\sigma}_{(0)11}^2,\boldsymbol{\sigma}_{(0)12},\cdots,\boldsymbol{\sigma}_{(0)1n},\boldsymbol{\sigma}_{(0)22}^2,\cdots,\boldsymbol{\sigma}_{(0)2n}$, $\boldsymbol{\sigma}_{(0)33}^2,\cdots,\boldsymbol{\sigma}_{(0)nn}^2$ are estimated.

In general, the expectation of weighted sum of the residuals squared is statistically satisfied with

$$\boldsymbol{E}(\boldsymbol{v}^T\boldsymbol{P}\boldsymbol{v})=\boldsymbol{tr}((\boldsymbol{Q}_{vv}\boldsymbol{P})^T\boldsymbol{P}\boldsymbol{\Sigma}_{ll}) \tag{9}$$

where $\boldsymbol{v}$ is the residual vector of $\boldsymbol{l}$, respectively, and $\boldsymbol{Q}_{vv}$ is the cofactor matrix of $\boldsymbol{v}$. On the ground of (9), one can appropriately construct suitable algorithms for variance and covariance estimation through the incorporation with the objective situations. The rigorous VC estimation algorithm after Helmert can be found in [Förstner, 1979; Kock, 1986; Cui et al, 2001; etc.]. The widely used simplification for the variance component estimation is given by [Förstner, 1979; etc.]

$$\hat{\boldsymbol{\sigma}}_{0i}^2=\frac{\boldsymbol{v}_i^T\boldsymbol{P}_{ii}\boldsymbol{v}_i}{\boldsymbol{n}_i-\boldsymbol{tr}\left(\boldsymbol{N}^{-1}\boldsymbol{B}_i^T\boldsymbol{P}_{ii}\boldsymbol{B}_i\right)} \tag{10}$$

for the independent measurement groups ($\boldsymbol{i}=1,\cdots,\boldsymbol{i},\cdots,\boldsymbol{m}$), where $\boldsymbol{n}_i$, $\boldsymbol{v}_i$, $\boldsymbol{P}_{ii}$ and $\boldsymbol{B}_i$ are the number, the residual vector, the weight matrix and the design matrix of the measurements in $\boldsymbol{i}$th group, respectively. In (10), the numerator is decomposed into the $\boldsymbol{m}$ sets of measurements whilst the denominator represents the redundancy index of the group of the measurements:

$$\boldsymbol{r}_i = \boldsymbol{trace}\left(\boldsymbol{P}_{ii}\boldsymbol{Q}_{v_i v_i}\right) = \boldsymbol{n}_i - \boldsymbol{tr}\left(\boldsymbol{N}^{-1}\boldsymbol{B}_i^T\boldsymbol{P}_{ii}\boldsymbol{B}_i\right) \tag{11}$$

If $\boldsymbol{n}_i = 1$, (10) becomes

$$\hat{\boldsymbol{\sigma}}_{0i}^2 = \frac{\boldsymbol{p}_{ii}\boldsymbol{v}_i^2}{(\boldsymbol{PQ}_{vv})_{ii}} = \frac{\boldsymbol{p}_{ii}\boldsymbol{v}_i^2}{1-\left(\boldsymbol{N}^{-1}\boldsymbol{b}_i^T\boldsymbol{b}_i\boldsymbol{p}_i\right)_{ii}} \tag{12}$$

for posteriori estimation of the variance factors of the individual independent measurements. The covariance factors may be estimated together with the variances [Förstner, 1979; Koch, 1986; Ou, 1989; etc.]. An approximate iterated method was given by [Ou, 1989]

$$\hat{\boldsymbol{\sigma}}_{(0)ii}^2 = \frac{\boldsymbol{v}_i^2\boldsymbol{p}_{ii}}{\boldsymbol{tr}((\boldsymbol{Q}_{vv}\boldsymbol{P})^T\boldsymbol{P}_{ii}\boldsymbol{Q}_{ii})} \tag{13}$$

for the covariance factors and

$$\hat{\boldsymbol{\sigma}}_{(0)ij} = \frac{2\boldsymbol{p}_{jk}\boldsymbol{v}_i^T\boldsymbol{v}_j}{\boldsymbol{tr}((\boldsymbol{Q}_{vv}\boldsymbol{P})^T\boldsymbol{P}_{ij}\boldsymbol{Q}_{ji}) + \boldsymbol{tr}((\boldsymbol{Q}_{vv}\boldsymbol{P})^T\boldsymbol{P}_{ji}\boldsymbol{Q}_{ij})} \tag{14}$$

for the covariance factors, where $\boldsymbol{Q}_{vv}$ is the cofactor matrix of $\boldsymbol{L}$ and

$$\boldsymbol{P}_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \mathrm{p}_{ii} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \text{ or } = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & \mathrm{p}_{ij} & \vdots \\ 0 & \dots & 0 & \dots & 0 \\ \vdots & \mathrm{p}_{ji} & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \tag{15}$$

$$\boldsymbol{Q}_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \mathrm{q}_{ii} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \text{ or } = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & \mathrm{q}_{ij} & \vdots \\ 0 & \dots & 0 & \dots & 0 \\ \vdots & \mathrm{q}_{ij} & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \tag{16}$$

The solution in (13) and (14) is based on the fact that the non-diagonal elements of the derived equation system from (9) are much small than its diagonal ones [Förstner, 1979; Ou, 1989; Wang et al, 2010]. The quantity $\boldsymbol{tr}(\boldsymbol{Q}_{vv}\boldsymbol{P})$ is identical with the total redundancy number r of the system (1). At most $r(r+1)/2$ independent variance and covariance components can be uniquely determined [Xu et al, 2007].

In practice, (12) ~ (15) may further be reformulated to fit to the specific applications. For example, the further combination of specific multiple variance or covariance factors under the reasonable assumption that they are identical based on the nature of the observation reality will definitely simplify these equations and improve their convergence by reducing the number of the estimated variance and covariance factors.

2.2. VC ESTIMATION IN SEQUENTIAL LEAST SQUARES

This subsection extends the VC component estimation to the sequential least squares. Assume to have a measurement sequence $\boldsymbol{L}_1,...,\boldsymbol{L}_k$ from epoch 1 to epoch $\boldsymbol{k}$. The observation equation system at $\boldsymbol{k}$ is given by:

$$\begin{pmatrix} \boldsymbol{L}_{\boldsymbol{x}_k} \\ \boldsymbol{L}_k \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_k \\ \boldsymbol{F}(\boldsymbol{x}_k) \end{pmatrix} + \begin{pmatrix} \boldsymbol{\Delta}_{L_x} \\ \boldsymbol{\Delta}_{L_k} \end{pmatrix} \tag{17}$$

where $\boldsymbol{L}_{\boldsymbol{x}_k} = \hat{\boldsymbol{x}}_{k-1} \sim \boldsymbol{N}(\hat{\boldsymbol{x}}_{k-1}, \boldsymbol{D}_{\hat{\boldsymbol{x}}_{k-1}\hat{\boldsymbol{x}}_{k-1}})$ is the current available estimated parameter vector $\boldsymbol{x}$ as the pseudo-observation with its accuracy $\boldsymbol{D}_{\hat{\boldsymbol{x}}_{k-1}\hat{\boldsymbol{x}}_{k-1}}$ from epoch $\boldsymbol{k}-1$ and $\boldsymbol{L}_k \sim \boldsymbol{N}(\tilde{\boldsymbol{L}}_k, \boldsymbol{D}_{L_k L_k})$ is the available observation at the current epoch with its expectation $\tilde{\boldsymbol{L}}_k$ and the variance matrix $\boldsymbol{D}_{L_k L_k}$. For simplicity, one can chose $\boldsymbol{L}_{\boldsymbol{x}_k}^{(0)} = \boldsymbol{x}^{(0)} = \hat{\boldsymbol{x}}_{k-1}$ in (17). Accordingly, one obtains the residual equation system for (17)

$$\begin{pmatrix} \boldsymbol{v}_{L_{x_k}} \\ \boldsymbol{v}_{L_k} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{B}_k \end{pmatrix} \boldsymbol{\delta\hat{x}}_k + \begin{pmatrix} \hat{\boldsymbol{x}}_{k-1} \\ \boldsymbol{F}(\hat{\boldsymbol{x}}_{k-1}) \end{pmatrix} \tag{18}$$

wherein $\boldsymbol{B}_k$ consists of the partial derivatives with respect to $\boldsymbol{x}$. The least squares solution for (18) states

$$\boldsymbol{\delta\hat{x}}_k = \boldsymbol{N}_k^{-1}(\boldsymbol{P}_{L_{x_k}}\hat{\boldsymbol{x}}_{k-1} + \boldsymbol{B}_k^T\boldsymbol{P}_k\boldsymbol{l}_k) \tag{19}$$

with its accuracy

$$\boldsymbol{D}_{\hat{x}_k\hat{x}_k} = \hat{\boldsymbol{\sigma}}_0^2\boldsymbol{N}_k^{-1} \tag{20}$$

wherein

$$\boldsymbol{N}_k = \boldsymbol{P}_{\hat{x}_{k-1}\hat{x}_{k-1}} + \boldsymbol{B}_k^T\boldsymbol{P}_k\boldsymbol{B}_k \tag{21}$$

$$\hat{\boldsymbol{\sigma}}_0^2 = \frac{\boldsymbol{v}_{L_k}^T\boldsymbol{P}_{L_kL_k}\boldsymbol{v}_{L_k} + \boldsymbol{v}_{L_{x_k}}^T\boldsymbol{P}_{L_{x_k}L_{x_k}}\boldsymbol{v}_{L_{x_k}}}{\boldsymbol{n}} \tag{22}$$

$$\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_{k-1} + \boldsymbol{\delta\hat{x}}_k \tag{23}$$

$$\boldsymbol{l}_k = \boldsymbol{L}_k - \boldsymbol{F}(\boldsymbol{x}^{(o)}) \tag{24}$$

$$\boldsymbol{P}_{L_{x_k}} = \boldsymbol{P}_{\hat{x}_{k-1}\hat{x}_{k-1}} = \boldsymbol{D}_{\hat{x}_{k-1}\hat{x}_{k-1}}^{-1} \tag{25}$$

$$\boldsymbol{P}_k = \boldsymbol{D}_{L_kL_k}^{-1} \tag{26}$$

In order to estimate the variance and covariance factors in $\boldsymbol{D}_{L_k L_k}$, they are first grouped sequentially as required by the application. And then, one can extend (13) and (14) to fit to multiple epochs in sequential least squares. Without giving the detailed derivation, the accumulated variance factors at epoch $\boldsymbol{k}$ is computed by

$$\hat{\sigma}^2_{(0)ii}(\boldsymbol{k}) = \frac{\sum_{m=1}^{k} \boldsymbol{v}^T_{i(m)} \boldsymbol{P}_{ii(m)} \boldsymbol{v}_{i(m)}}{\sum_{m=1}^{k} tr((\boldsymbol{Q}_{vv(m)} \boldsymbol{P}_{(m)})^T \boldsymbol{P}_{ii(m)} \boldsymbol{Q}_{ii(m)})} \tag{27}$$

and the covariance factors by

$$\hat{\sigma}_{(0)ij}(\boldsymbol{k}) = \frac{2\sum_{m=1}^{k} \boldsymbol{v}^T_{i(m)} \boldsymbol{P}_{ij(m)} \boldsymbol{v}_{j(m)}}{\sum_{m=1}^{k} \left[tr((\boldsymbol{Q}_{vv(m)} \boldsymbol{P}_{(m)})^T \boldsymbol{P}_{ij(m)} \boldsymbol{Q}_{ji(m)}) + tr((\boldsymbol{Q}_{vv(m)} \boldsymbol{P}_{(m)})^T \boldsymbol{P}_{ji(m)} \boldsymbol{Q}_{ij(m)}) \right]} \tag{28}$$

where $\boldsymbol{v}_{i(m)}$ is the measurement residual vector of the $\boldsymbol{i}$th measurement group at epoch $\boldsymbol{m}$.

2.3. Static Relative GPS Positioning

The objective of static relative position is to determine the coordinates of an unknown point $\boldsymbol{A}$ with respect to a known point $\boldsymbol{B}$, or the baseline between them, both of which are stationary. This type of positioning requires simultaneous observations at two points and can be performed with code ranges or/and carrier phases (Hofmann-Wellenhof et al., 2008). Double-differenced measurements are usually used and the measurement equation for a single frequency receiver is given by:

$$\Delta\nabla \boldsymbol{P}^{jk}_{AB} = \Delta\nabla \rho^{jk}_{AB} + \Delta\nabla \varepsilon^{jk}_{P,AB} \tag{29}$$

$$\lambda_1 \Delta\nabla \phi^{jk}_{AB} = \Delta\nabla \rho^{jk}_{AB} + \lambda_1 \Delta\nabla \boldsymbol{N}_1 + \Delta\nabla \varepsilon^{jk}_{L1,AB} \tag{30}$$

where $\Delta\nabla \boldsymbol{P}^{jk}_{AB}$ is the double differenced L1 code range measurement vector between satellite $\boldsymbol{j}$ and satellite $\boldsymbol{k}$ and between receiver $\boldsymbol{A}$ and receiver $\boldsymbol{B}$ in meters, $\Delta\nabla \phi^{jk}_{AB}$ is the corresponding L1 double differenced carrier phase measurement vector in cycles, $\Delta\nabla \rho^{jk}_{AB}$ is the geometric range, λ_1 is the wave length of L1 carrier phase, $\nabla\Delta \boldsymbol{N}_1$ is the double differenced ambiguity unknown vector of L1 carrier phases, $\Delta\nabla \varepsilon^{jk}_{P,AB}$ is the code measurement noise vector and $\Delta\nabla \varepsilon^{jk}_{L1,AB}$ is the phase measurement noise vector, respectively. For short baselines, the residual tropospheric and ionospheric effects in double differenced measurements are assumed to be negligibly small and are ignored. The variance-covariance matrix for the double differenced measurements in (29) and (30) is given as follows:

$$\boldsymbol{D}_{\Delta\nabla} = \begin{pmatrix} \sigma^2_{A,j} + \sigma^2_{B,j} + \sigma^2_{A,1} + \sigma^2_{B,1} & \sigma^2_{A,j} + \sigma^2_{B,j} & \cdots & \sigma^2_{A,j} + \sigma^2_{B,j} \\ \sigma^2_{A,j} + \sigma^2_{B,j} & \sigma^2_{A,j} + \sigma^2_{B,j} + \sigma^2_{A,2} + \sigma^2_{B,2} & \cdots & \sigma^2_{A,j} + \sigma^2_{B,j} \\ \vdots & \vdots & & \vdots \\ \sigma^2_{A,j} + \sigma^2_{B,j} & \sigma^2_{A,j} + \sigma^2_{B,j} & \cdots & \sigma^2_{A,j} + \sigma^2_{B,j} + \sigma^2_{A,n} + \sigma^2_{B,n} \end{pmatrix} \tag{31}$$

where $\sigma^2_{A,j}$ and $\sigma^2_{B,j}$ are the variances for the measurements to the reference satellite $\boldsymbol{j}$ while $\sigma^2_{A,k}$ and $\sigma^2_{B,k}$ for $\boldsymbol{k} = 1,...,\boldsymbol{n}$ are the variances for the measurements to the remaining satellites. The algorithm can only estimate the variance and covariance factors for each entry in the matrix. Therefore, the individual variances cannot be separated. Thus, (31) can be written as

$$\boldsymbol{D}_{\Delta\nabla} = \begin{pmatrix} \sigma^2_1 & \sigma^2_j & \cdots & \sigma^2_j \\ \sigma^2_j & \sigma^2_2 & \cdots & \sigma^2_j \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2_j & \sigma^2_j & \cdots & \sigma^2_n \end{pmatrix} \tag{32}$$

For each epoch there are a maximum of $\boldsymbol{n}+1$ variances factors that can be estimated because all of the non-diagonal factors stand for the identical variance factors $\sigma^2_{A,j} + \sigma^2_{B,j}$.

Furthermore, the elevation angle of a satellite with respect to the user location influences the signal path length and the strength of the received signal (Hofmann-Wellenhof et al., 2008). As the elevation of the satellites decreases, the signal-to-noise ratio increases. In the VC estimation algorithm the measurements can accordingly be grouped based on the elevation angles.

3. NUMERICAL EXAMPLES

The algorithm described in section 2 was implemented for GPS relative positioning in static mode. With both of the simulated and real data, double differenced L1 C/A and carrier phase measurements were processed. The integer ambiguities were resolved using the LAMBDA method (Teunissen et al, 1997). The initial variances assigned to code and phase measurements were arbitrary and independent of satellite elevation angles. The variance-covariance components were grouped against satellite elevations. It assumes that the variance does not experience significant change with limited changes of the satellite elevation angle. The ranges of the satellite elevation angles for each group reasonably vary so that each group can have almost the same number of measurements. This ensures that there are enough residuals to estimate all of the variance components reliably. As a result there will be uneven intervals and also be different for each dataset.

3.1. Simulated Data

The undifferenced code and phase static measurements were simulated at 1.0 Hz for 3900 seconds with an elevation mask of 10 degrees. The measurement noise variance applied to the measurements varies with the satellite elevation angle based on

$$\boldsymbol{\sigma}_E^2 = \boldsymbol{\sigma}_{90}^2 [0.5 + 0.5\mathbf{exp}(17.5/\boldsymbol{E})] \tag{33}$$

where $\boldsymbol{\sigma}_{90}^2$ is the variance at the zenith and E is the satellite elevation angle in degrees. The zenith variances used for code and phase measurements are $(0.300\boldsymbol{m})^2$ and $(0.003\boldsymbol{m})^2$ respectively. The initial variances for double differenced code and phase measurements used in this test were $(0.800\boldsymbol{m})^2$ and $(0.016\boldsymbol{m})^2$, respectively. Fig. 1 shows the covariance estimation as a function of time together with its true value. The results show that the covariance component for the code and phase measurement can be estimated with at least 120 and 240 epochs respectively.

Fig. 2 and 3 show the estimated standard deviations as a function of time together with its true value. The results from three measurement groups are presented: the top, middle and bottom contain the highest, average and lowest elevation angles respectively.

The estimation of the second group (Elevation 33 to 35) starts at 1510 seconds, which is the time when one or more satellites enter this elevation range. Similarly the third group starts at 150 seconds. The results show that the proposed algorithm takes at least 85 epochs and 135 to estimate the code and phase standard deviations. Fig. 4 shows the estimated standard deviation for all the groups as a function of satellite elevation angles together with it corresponding initial and true value plots. All the estimated components fit closely to the true curve.

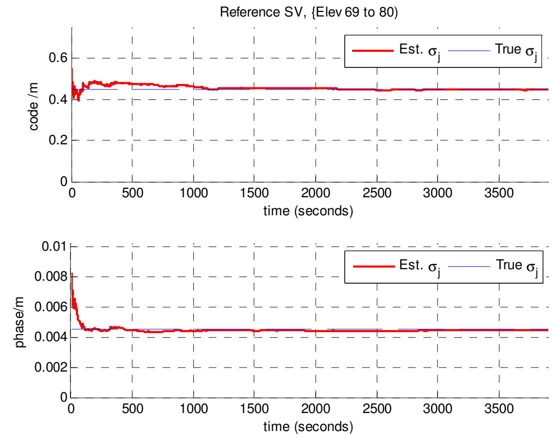

Fig. 1 Code and phase Ref. SV σ_j vs. time

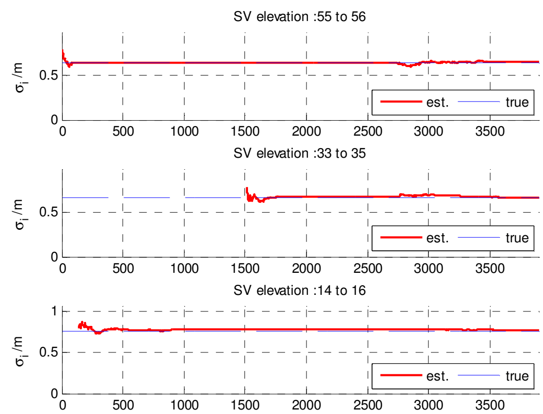

Fig. 2 Selected code standard deviation vs. time

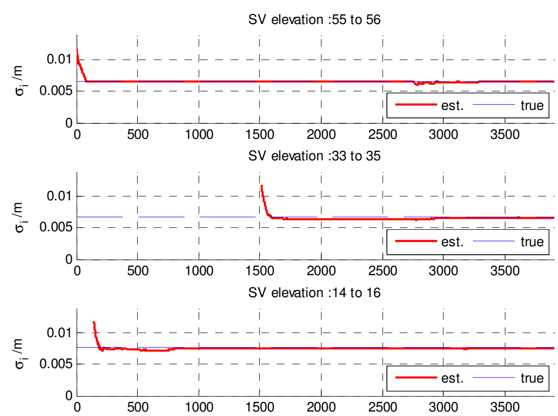

Fig. 3 Selected phase standard deviation vs. time

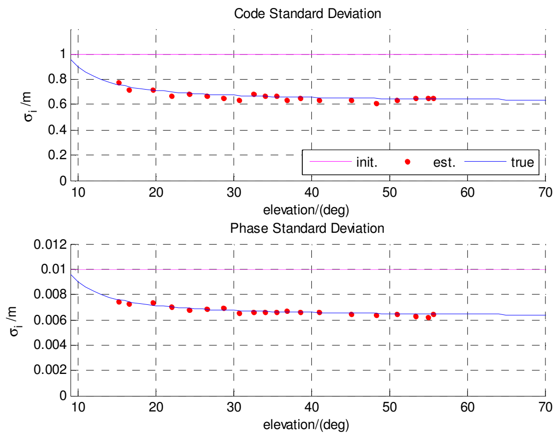

Fig. 4 Code and phase standard deviation vs. satellite elevation

Fig. 5 and 6 show the position errors together with the corresponding standard deviation for both without and with VC estimation.

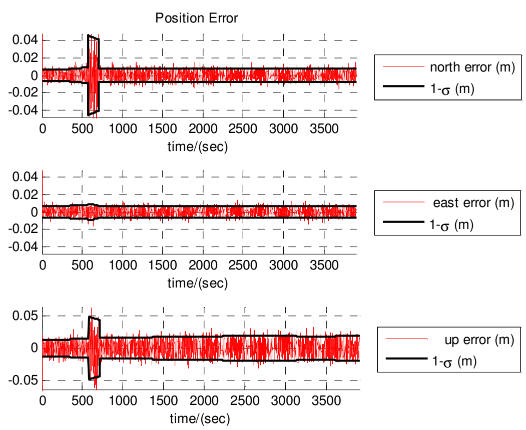

Fig. 5 Position error without VC estimation

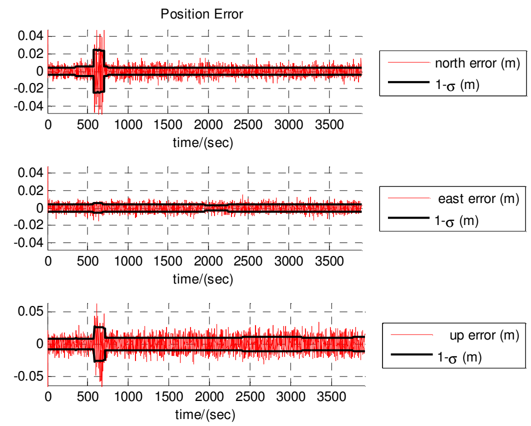

Fig. 6 Position error with VC estimation

Fig. 7 and 8 show the probability density functions (PDFs) of standardized position error without and with VC estimation together with a standardized normal distribution curve. The PDFs with the VC estimation algorithm fits the standardized normal distribution curve which suggests that the estimated variances for the unknowns are realistic.

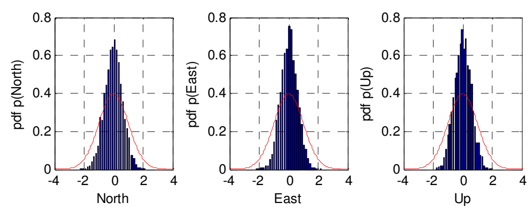

Fig. 7 PDFs of standardized position error using initial variances (without VC estimation)

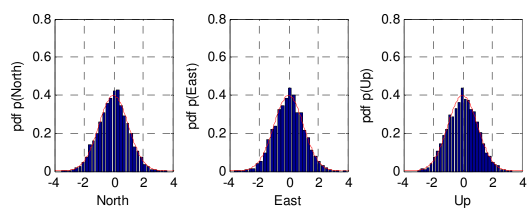

Fig. 8 PDFs of standardized position error with VC estimation

3.2. Real Data

This section presents the processing results using the VC estimation algorithm on real GPS data. The static data was collected on 24 April 2010 using two Leica 1200 receivers for the duration of 65 minutes. The receivers were set at sampling rate of 1.0 Hz and an elevation mask of 10 degrees. The base station was located at N43°46'26.34512", W79°30'43.23784", 158.697 m in

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York University, Toronto, Ontario, Canada. The baseline length is 63.395m. The mean of the estimated position was used as the 'true' rover position.

The double differenced code and phase variances used in this test are $(0.800\boldsymbol{m})^2$ and $(0.016\boldsymbol{m})^2$. Fig. 9 shows the covariance estimation as a function of time.

Fig. 10 and 11 shows the standard deviation estimation as a function of time together. The results from three measurement groups are presented: the top, middle and bottom contain the highest, average and lowest elevation angles respectively.

With the real data it takes at least 170 epochs and 200 to estimate the code and phase standard deviation. In fig 8 first plot, the constant line between 50 and 2750 seconds indicates there are no measurements between elevations 55 and 56. Similar features can be found in other plots. Fig. 12 shows the estimated standard deviation for all the groups as a function of satellite elevation angles together with the initial value plot.

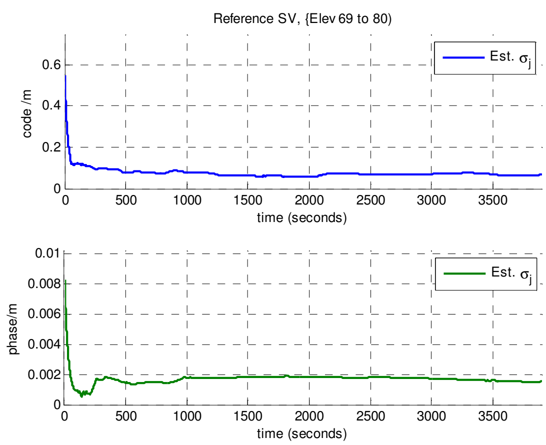

Fig. 9 Code and phase covariance vs. time

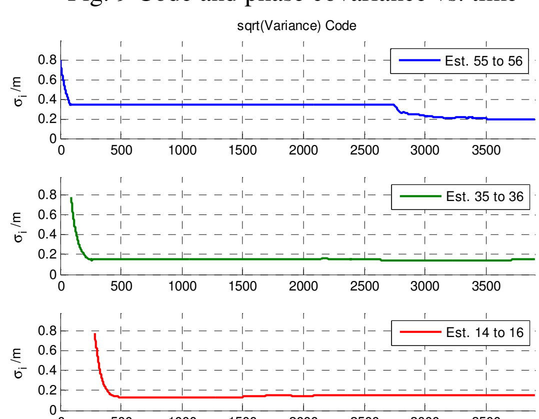

Fig. 10 Selected code standard deviation vs. time

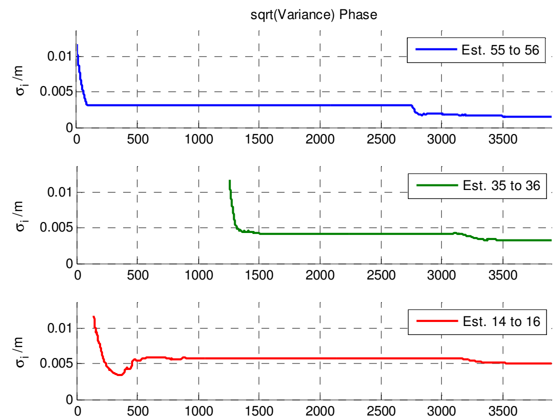

Fig. 11 Selected phase standard deviation vs. time

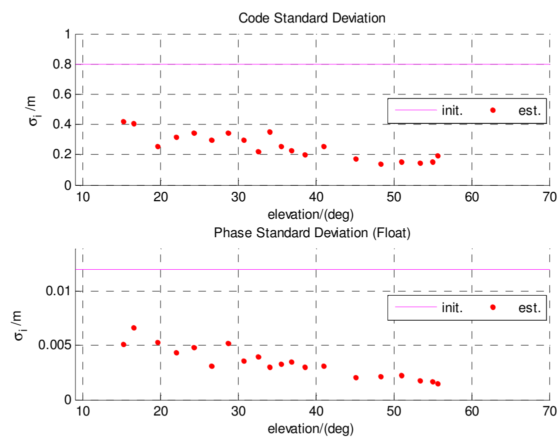

Fig. 12 Code and phase variances vs. satellite elevation

As expected the estimated variances generally decreases as satellite elevation increases. Fig. 13 and 14 show the position errors without and with VC estimation.

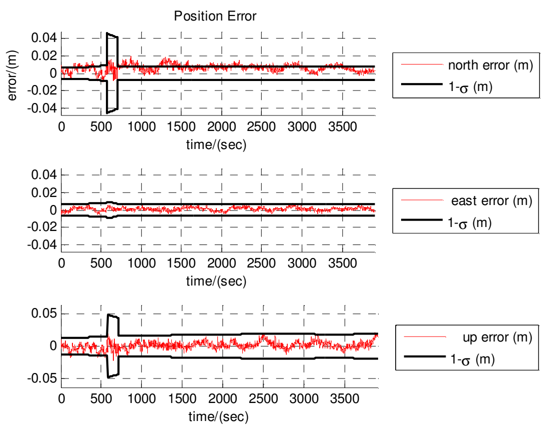

Fig. 13 Position Error without VC estimation

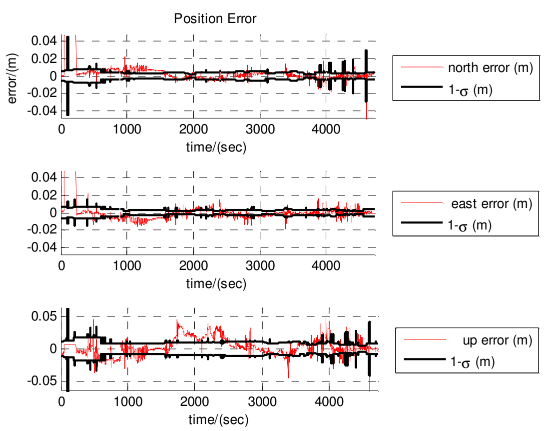

Fig. 14 Position Error with VC estimation

Fig. 15 and 16 show the PDFs of standardized position error without and with VC estimation together with a standardized normal distribution curve. The PDFs with the VC estimation algorithm show that the there is a bias in the position solution with unity variance.

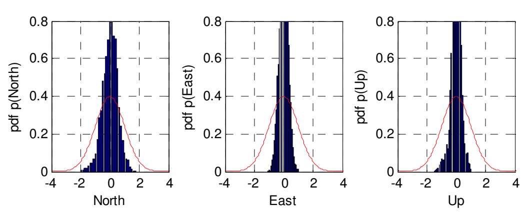

Fig. 15 PDFs of standardized position error using initial variances

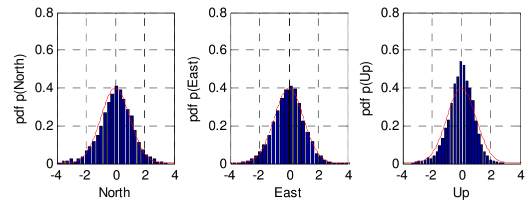

Fig. 16 PDFs of standardized position error after VC estimation

4. REMARKS AND CONCLUSIONS

The paper presents a VC estimation algorithm for the sequential least squares methods with correlated measurements. The results show that the posteriori accuracy of the unknowns is more realistic with the VC estimation. Furthermore the estimated variance and covariance have good convergence. The results from the simulated data show that both variance and covariance elements are estimated accurately. The results from simulated and real datasets show the feasibility, efficiency and practicality of the algorithm.

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