A Model for Analysis of Drug Abuse and HIV Infection

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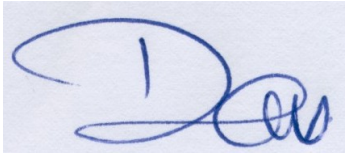
Abstract

The project sought to investigate the effects of drug abuse on the progression of HIV infection. We capture, in the model, the dynamics of drug users and non-drug users and investigate their impacts in HIV transmission. Mathematical analysis showed that the model has six equilibrium points where one of the equilibrium points is a disease-free equilibrium point and the rest are endemic equilibrium points which represented various hypothetical scenarios that influence the progression of HIV infection. Results of the Numerical simulations showed that the situation where both non-drug users and drug users are contributing to HIV infection worsens the progression of HIV infection.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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1. Introduction

The Human Immunodeficiency Virus (HIV) is a retrovirus that attacks the human immune system making it unable to fight against HIV infection and other infections. The continuous deterioration of the strength of the immune system leads to the development of Acquired Immunodeficiency Syndrome (AIDS) which ultimately leads to the death of individuals (Ngwenya, 2009; US13). HIV is transmitted through unprotected sexual intercourse, mother-to-child transmission (breast feeding), blood transfusion, unsterilised body piercing or tattooing and through sharing of needles and syringes (KGcount).

HIV infection is one of the sexually transmitted infections which is a world pandemic (Hethcote, 2000). An HIV infected individual can stay with the infection for some years without getting ill or showing any symptoms of it (US13). There are four stages of progression from initial HIV infection to the AIDS stage. These stages are the primary, chronic, pre-AIDS and AIDS stages (WHO05). The primary HIV infection lasts for a few weeks and after which is often accompanied by short flu-like illness. During this stage, there is a large amount of HIV in the peripheral blood and the immune system begins to respond to the virus by producing HIV antibodies and cytotoxic lymphocytes. This process of developing detectable antibodies due to HIV infection is called seroconversion. If an HIV antibody test is done before the completion of seroconversion, then the test may not be positive. The chronic stage lasts for an average of ten years. At this stage the level of HIV in the peripheral blood drops to very low levels but people remain infectious and HIV antibodies are detectable in the blood. In this case, HIV antibody tests will show a positive result. The pre-AIDS stage is characterised by the immune system getting severely damaged by HIV due to the failure by the body to repair the lymph nodes and tissues, and, HIV mutating and becoming stronger. This stage is mainly associated with the emergence of opportunistic infections such as tuberculosis, cancer, malaria etc. The AIDS stage is characterized by extensive damage to the immune system and development of increasingly severe opportunistic infections and cancers (WHO05).

A drug is a chemical that can alter a person's perceptions, feelings, behaviours or thoughts. Drug abuse can be defined as a disorder characterised by a descriptive pattern of using a substance that leads to significant problems or distress involving one to tolerate or withdraw from the substance.Drug abuse can also be called substance abuse or chemical abuse. All substances whose ingestion can lead to a high feeling can be abused (Web13). Examples of illegal drugs abused are marijuana, cocaine, amphetamines, anabolic steroid and nicotine among others (Web13). The effects of drug abuse ranges from development of seizures and death, paranoia to infertility and organ failure. Patterns of drug abuse vary from several times a day to infrequently, sometimes with excessive amounts. Individuals intoxicated with drugs have higher chances of engaging in unsafe sexually practices (UNHCR and WHO, 2008). Regular use of drug results in individuals craving to have more quantities to achieve the same effect and thus depend on the drug. Drug abuse and addiction have been closely linked with HIV/AIDS since the beginning of the epidemic (UNHCR and WHO, 2008).

Drug abuse and addiction can also worsen the progression of HIV and its consequences, especially in the brain (US13). Animal studies found that some drugs like Methamphetamine increased HIV viral replication. HIV infection among drug users is increased through injection drug use and poor judgement and risk behaviours. Injection drug use contributed 20% of new HIV diagnosis among women (Marcondes, 2010). Poor judgement and risky behaviour can lead to unsafe sex which puts people at risk of getting HIV or transmitting it to other people (Marcondes, 2010). Globally 5 to 10% of HIV infections result from injection drug users (WHO et al., 2004). Figure 1.1 shows the various

pathways people may go through under the influence of drug abuse on HIV.

A diagnosis of HIV or AIDS on a drug user can precipitate an increase in or relapse into substance abuse, but it can also lead to positive behavioural change. Some people start to use drugs to relieve themselves of difficulties, painful feelings and stressful situations and when they find out that they are HIV positive, they may feel compelled to turn to drugs once again to help themselves cope with this diagnosis and its implications. Other people may change positively when they are HIV positive. Treatment has been used as one of the successful intervention strategies for HIV and drug abuse. Drug abusers on treatment may reduce their drug intake and related risk behaviours like drug injection and unsafe sex. HIV treatment with antiretroviral drugs reduces the level of infectiousness of individuals as well as the rate of transmission of HIV. Other intervention strategies include counselling, education, nutrition and abstinence among others (WebTD).

The interactions between HIV and drug abuse are complex and in this project we want to investigate the effects of drug abuse on HIV infection progression using mathematical models. The first chapter so far has given the basic background information on HIV and drug abuse. In chapter 2, we will review some studies done on HIV and drug abuse and use these as a basis of our model formulation. In chapter 3, we will present a model for HIV and drug abuse and analyse it. Numerical simulations will be done and results discussed in chapter 4.

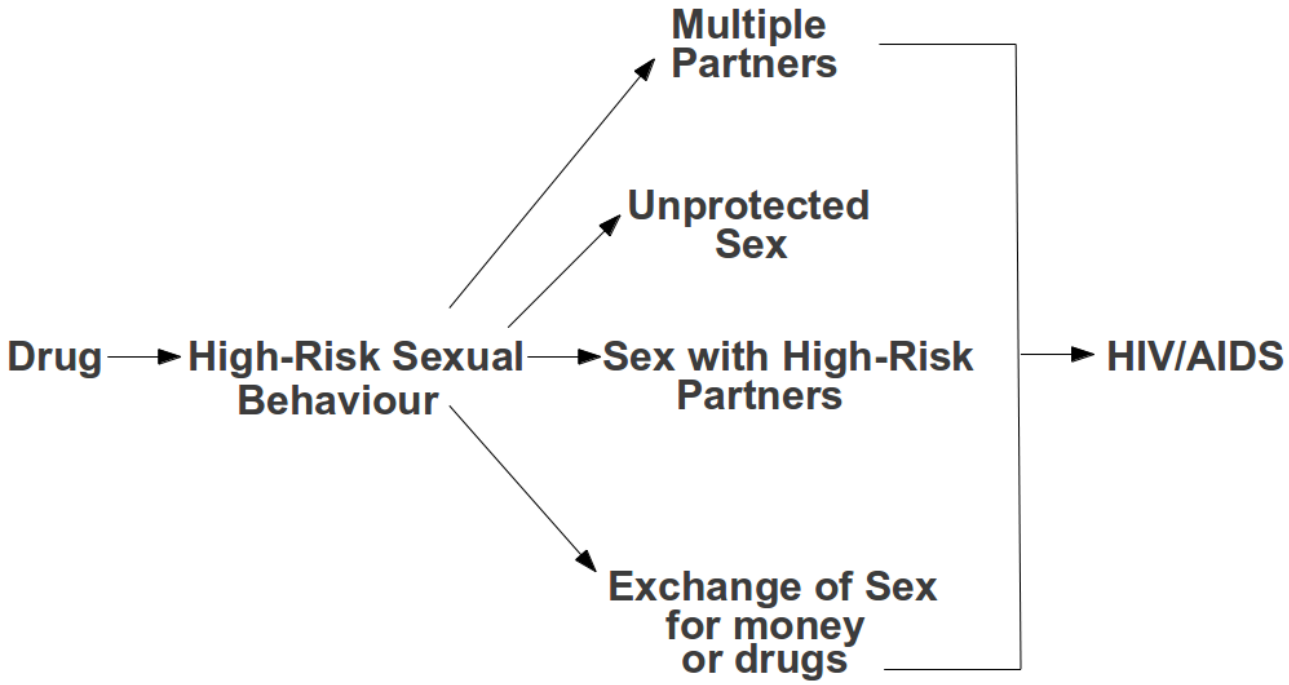

Figure 1.1: The diagram of the influence of drug abuse on HIV (Bloom)

2. Literature Review

In this chapter we review 3 selected studies done either on drug abuse, HIV infection or HIV and drug abuse and use these studies as building blocks to our study on drug abuse and HIV infection.

Rossi (Rossi, 2002) studied the role of dynamic modelling in drug abuse epidemiology. The study was motivated by the need on policy makers for information to describe and understand drug use and design appropriate intervention. Under-reporting of drug abuse data motivated the use of mathematical models where the extent of the phenomenon was estimated and dynamics observed from existing statistical and documentary information on drug abuse. In the study, dynamical models were used to generate estimates or to verify hypothesis or predict drug abuse trends. The study used models of epidemics effectively in the drug field to provide evidence for public health-oriented interventions and policies. The model provided a framework in which numbers of people in different compartments and the relationships between such compartments were described in mathematical terms. Qualitative and quantitative analyses were presented to demonstrate the potential usefulness of the model for decision makers on the data for Italy on the heroin epidemic. Results of the model suggested that the spread of the infectious disease epidemic among injecting drug users correlated to the hidden part of the drug user's career and that interventions must be aimed on reducing latency period. The study only focussed on fitting the model to data but did not provide a full mathematical analysis of the model.

Baryarama et al. (2005) formulated an HIV/AIDS model which incorporated complacency for the adult population. The study investigated the effects of adult individuals returning to high risk sexual behaviours. A 3-state compartmental model was formulated which captured the dynamics of the susceptible individuals, HIV infected individuals and AIDS individuals. Detailed mathematical analysis was carried out together with the numerical simulations and showed that complacency, which resulted from dependence of HIV transmission on the number of AIDS cases in a community, led to damped periodic oscillations in the number of infective individuals with oscillations more marked at lower rates of progression to AIDS. The implications of these results on public health with respect to monitoring the HIV/AIDS epidemic and widespread use of antiretroviral (ARV) drugs was discussed. The results showed a tendency for the epidemic to stabilize at higher numbers of infective and AIDS cases than the minimum numbers attained during the first decline of the epidemic. The study showed that prolonging survival of AIDS cases may lower the endemic equilibrium level of HIV infection.

Kalula and Nyabadza (Kalula and Nyabadza, 2012) studied a theoretical model for substance abuse in the presence of treatment in the South African context. Their study used a six-state mathematical model to investigate qualitatively the dynamics of drug abuse and predict the drug abuse trends. The mathematical analysis of the model on substance abuse epidemic revealed an important threshold parameter called the basic reproduction number which was used as a measure of secondary cases that an infected individual produced in a completely susceptible population. The model was fitted to data using the least squares curve fitting method on methamphetamine users in Western Cape who entered rehabilitation . Results of the mathematical analysis showed that the proposed model had a backward bifurcation which indicated the existence of complex dynamics in drug abuse. The backward bifurcation resulted from introduction of treatment of drug abuse in the model. This meant that it is not sufficient to reduce the basic reproduction number below unit to control the substance abuse epidemic when treatment was used as intervention. The results also suggested that the substance abuse epidemic can be reduced by intervention programmes targeted at light drug users and by increasing the uptake rate into treatment for those addicted. Their results also showed that the spread of substance abuse could be controlled through a reduction in the relapse rate, increasing interventions

at light drug users phase and increasing the uptake rates into treatment.

We use ideas from these studies to formulate a mathematical model to investigate implications of drug abuse and HIV infection on the progression of HIV infection.

3. Mathematical Analysis of the model

3.1 Model Formulation

In this chapter, we formulate the model incorporating HIV infection and drug abuse. The variables and parameters used in our model are defined in table 3.1. We assume that individuals are recruited into the population through birth and immigration at a constant rate π, and can only participate in the HIV and drug abuse dynamics when they are sexually active. A proportion ν of these recruited individuals will not use drugs and $1-\nu \quad (0 \leq \nu \leq 1)$ will become addicted to the drugs. Individuals not using drugs (non-drug users) in compartment S will die naturally at a constant rate μ. They get infected with HIV at a rate $\lambda_N S$ where λ_N is a force of infection defined by $\lambda_N = \beta(I + \eta_2 A)$ where β and η_2 are defined in table 3.1. A force of infection is the probability that a susceptible will get HIV per unit time. Drug users in compartment U die naturally at a constant rate μ or die due to the effects of drug abuse at a constant rate δ_1. Drug users get infected with HIV at a rate $\lambda_D U$, where λ_D is the force of infection defined by $\lambda_D = \beta(\eta_1 I_u + \eta_3 A_u)$ where η_1 and η_3 are defined in table 3.1. Susceptible individuals infected with HIV move to the HIV infected non-drug users compartment I. The infected non-drug users leave the compartment through natural death at constant rate μ or through progression to the non-drug users AIDS compartment A at a constant rate ρ_1. Drug users who get infected with HIV move to the compartment of HIV infected drug users, I_u. The HIV infected drug users leave the compartment either through natural death, death due to drug abuse (at a constant rate δ_1) or through progression to the drug-users AIDS compartment A_u. Individuals in both AIDS compartments are removed through natural death or death due to AIDS. We assume that AIDS drug users are most likely to die due to AIDS other than drug abuse effects since they are mostly inactive and have little or no chance of engaging in activities that endanger their lives and due to the effects of drug abuse, $\eta_2 < \eta_3$ and $\rho_1 < \rho_2$. The schematic diagram for the model is given in figure 3.1. The model representing the dynamics of HIV and drug abuse is a system of non-linear ordinary differential equation given by equations (3.1) to (3.6).

$$\frac{dS}{dt} = \pi\nu - \lambda_N S - \mu S, \tag{3.1}$$

$$\frac{dU}{dt} = \pi(1-\nu) - \lambda_D U - (\mu + \delta_1)U, \tag{3.2}$$

$$\frac{dI}{dt} = \lambda_N S - (\mu + \rho_1)I, \tag{3.3}$$

$$\frac{dI_u}{dt} = \lambda_D U - (\mu + \rho_2 + \delta_1)I_u, \tag{3.4}$$

$$\frac{dA}{dt} = \rho_1 I - (\mu + \delta_2)A, \tag{3.5}$$

$$\frac{dA_u}{dt} = \rho_2 I_u - (\mu + \delta_2)A_u, \tag{3.6}$$

$$\lambda_N(t) = \beta(I + \eta_2 A) \tag{3.7}$$

$$\lambda_D(t) = \beta(\eta_1 I_u + \eta_3 A_u). \tag{3.8}$$

Table 3.1: Model variables, parameters and their descriptions

Variable	Description
$S(t)$	Population of susceptible non-drug users.
$U(t)$	Population of susceptible drug users.
$I(t)$	Population of infected non-drug users.
$I_u(t)$	Population of infected drug users.
$A(t)$	Population of AIDS non-drug users.
$A_u(t)$	Population of AIDS drug users.
Parameter	Description
π	Recruitment rate of susceptible individuals.
ν	Proportion of recruited non-drug users susceptible individuals
μ	Natural death rate.
ρ_1	Rate of progression to AIDS for infected non-drug users.
ρ_2	Rate of progression to AIDS for infected drug users.
δ_1	Death rate due to drug use.
δ_2	Death rate due to AIDS.
β	Effective contact rate.
$\eta_1 > 1$	Rate for level of risk of disease transmission by infected drug users.
$\eta_2 < 1$	Rate for level of risk of disease transmission by AIDS non-drug users.
$\eta_3 < 1$	Rate for level of risk of disease transmission by AIDS drug users.

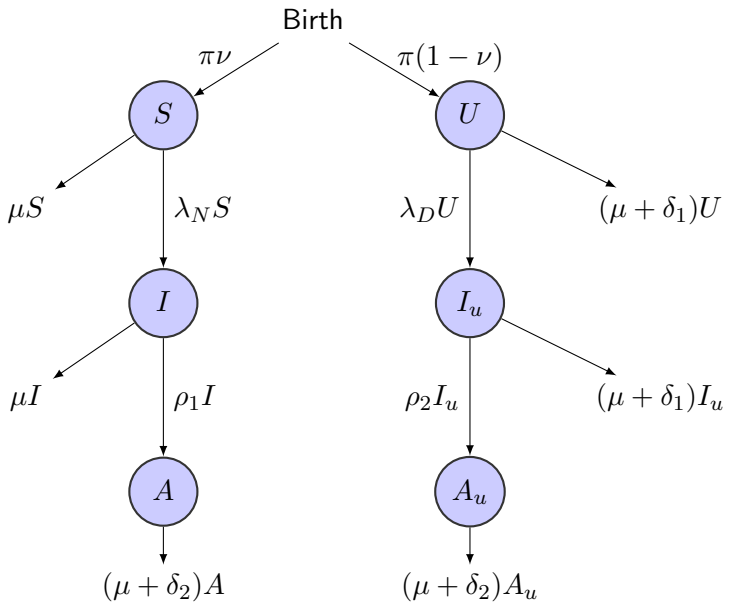

Figure 3.1: Model diagram

3.2 Model analysis

3.2.1 Disease free equilibrium point. The disease free-equilibrium is the point where there are no HIV infected and AIDS individuals.

At the disease free equilibrium, we have

$$I^* = I_u^* = A^* = A_u^* = 0. \tag{3.9}$$

Setting the right hand side of equation (3.1) to (3.6) and applying (3.9), we have

$$\pi\nu - \mu S^* = 0, \tag{3.10}$$

$$\pi(1-\nu) - (\mu + \delta_1)U^* = 0, \tag{3.11}$$

which gives $S^* = \dfrac{\pi\nu}{\mu}$ and $U^* = \dfrac{\pi(1-\nu)}{\mu+\delta_1}$.

The disease free equilibrium is given by

$$E_0 = \left(\frac{\pi\nu}{\mu}, \frac{\pi(1-\nu)}{\mu+\delta_1}, \quad 0, \quad 0, \quad 0, \quad 0\right).$$

We note that when $\nu = 1$, we have an HIV-free and drug-free equilibrium point. When $\nu = 0$, we have an HIV-free but not drug-free equilibrium point.

3.2.2 Reproduction Number $\mathcal{R}_0$. The basic reproduction number $\mathcal{R}_0$ is the number of secondary infections caused by an individual introduced into a totally susceptible population. We can distinguish new infections from all other changes in population in order to find $\mathcal{R}_0$ by letting $\mathcal{F}_i$ be the rate of appearance of new infections in compartment i, $\mathcal{V}_i^+(x)$ be the rate of transfer of individuals into compartment i by all other means, $\mathcal{V}_i^-(x)$ be the rate of transfer of individuals out of compartment i and $\mathcal{R}_0 = \rho(FV^{-1})$ (van den Driessche and Watmough, 2002).

Since we have four infectious classes I, I_u, A, A_u, the matrix showing the rate of appearance of new infections in compartment i is given by

$$\mathcal{F} = \begin{pmatrix} \lambda_N S \\ \lambda_D U \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \\ \mathcal{F}_4 \end{pmatrix}.$$

The matrix showing the rate of transfer of individuals in and out of compartments i is

$$\mathcal{V} = \mathcal{V}^- - \mathcal{V}^+ = \begin{pmatrix} (\mu+\rho_1)I \\ (\mu+\rho_2+\delta_1)I_u \\ (\mu+\delta_2)A - \rho_1 I \\ (\mu+\delta_2)A_u - \rho_2 I_u \end{pmatrix} = \begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \end{pmatrix}$$

$$\text{where} \quad \mathcal{V}^+ = \begin{pmatrix} 0 \\ 0 \\ \rho_1 I \\ \rho_2 I_u \end{pmatrix} \quad \text{and} \quad \mathcal{V}^- = \begin{pmatrix} (\mu+\rho_1)I \\ (\mu+\rho_2+\delta_1)I_u \\ (\mu+\delta_2)A \\ (\mu+\delta_2)A_u \end{pmatrix}.$$

The jacobian matrix of $\mathcal{F}$ evaluated at the disease free equilibrium point is given by

$$F = \frac{\partial \mathcal{F}(E_0)}{\partial x_j} = \begin{pmatrix} \beta S_0 & 0 & \beta\eta_2 S_0 & 0 \\ 0 & \beta\eta_1 U_0 & 0 & \beta\eta_3 U_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where} \quad x_j = I, I_u, A, A_u \quad \text{for} \quad j = 1 \ldots 4,$$

S_0 and U_0 are respectively $\dfrac{\pi\nu}{\mu}$ and $\dfrac{\pi(1-\nu)}{\mu+\delta_1}$.

The jacobian matrix of $\mathcal{V}$ evaluated at the disease free equilibrium point E_0 is

$$V = \frac{\partial \mathcal{V}(E_0)}{\partial x_j} = \begin{pmatrix} (\mu+\rho_1) & 0 & 0 & 0 \\ 0 & (\mu+\rho_2+\delta_1) & 0 & 0 \\ -\rho_1 & 0 & (\mu+\delta_2) & 0 \\ 0 & -\rho_2 & 0 & (\mu+\delta_2) \end{pmatrix}.$$

The inverse of V is

$$V^{-1} = \begin{pmatrix} \dfrac{1}{(\mu+\rho_1)} & 0 & 0 & 0 \\ 0 & \dfrac{1}{(\mu+\rho_2+\delta_1)} & 0 & 0 \\ \dfrac{\rho_1}{(\mu+\rho_1)(\mu+\delta_2)} & 0 & \dfrac{1}{(\mu+\delta_2)} & 0 \\ 0 & \dfrac{\rho_2}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)} & 0 & \dfrac{1}{(\mu+\delta_2)} \end{pmatrix}.$$

The next generation matrix FV^{-1} is given by

$$\begin{pmatrix} \beta S_0 & 0 & \beta\eta_2 S_0 & 0 \\ 0 & \beta\eta_1 U_0 & 0 & \beta\eta_3 U_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{1}{(\mu+\rho_1)} & 0 & 0 & 0 \\ 0 & \dfrac{1}{(\mu+\rho_2+\delta_1)} & 0 & 0 \\ \dfrac{\rho_1}{(\mu+\rho_1)(\mu+\delta_2)} & 0 & \dfrac{1}{(\mu+\delta_2)} & 0 \\ 0 & \dfrac{\rho_2}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)} & 0 & \dfrac{1}{(\mu+\delta_2)} \end{pmatrix},$$

$$= \begin{pmatrix} A & 0 & B & 0 \\ 0 & C & 0 & D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} A &= \frac{\beta\rho_1\eta_2 S_0}{(\mu+\delta_2)(\mu+\rho_1)} + \frac{\beta S_0}{(\mu+\rho_1)} = \frac{\beta S_0}{(\mu+\rho_1)}\Big(\frac{\rho_1\eta_2+\mu+\delta_2}{(\mu+\delta_2)}\Big), \\ B &= \frac{\beta\eta_2 S_0}{(\mu+\delta_2)}, \\ C &= \frac{\beta\rho_2\eta_3 U_0}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)} + \frac{\beta\eta_1 U_0}{(\mu+\rho_2+\delta_1)} = \frac{\beta U_0}{(\mu+\rho_2+\delta_1)}\Big(\frac{\rho_2\eta_3+\eta_1(\mu+\delta_2)}{(\mu+\delta_2)}\Big), \\ D &= \frac{\beta\eta_3 U_0}{(\mu+\delta_2)}. \end{aligned}$$

The eigenvalues of FV^{-1} are obtained by calculating the roots of the characteristics equation of FV^{-1} as follows:

$$|FV^{-1} - \lambda I| = \begin{vmatrix} A-\lambda & 0 & B & 0 \\ 0 & C-\lambda & 0 & D \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0,$$

$$(A-\lambda)(C-\lambda)\lambda^2 = 0,$$

$$\lambda = 0,\ \lambda = A,\ \lambda = C.$$

Let $A = \mathcal{R}_h$ and $C = \mathcal{R}_d$.

The basic reproduction number of the system of equations (3.1) to (3.6) is the spectral radius of the matrix FV^{-1} denoted by $\rho(FV^{-1})$. Thus

$$\mathcal{R}_0 = \rho(FV^{-1}) = \max\{\mathcal{R}_h, \mathcal{R}_d\}.$$

$\mathcal{R}_h$ is the reproduction number due to non-drug users HIV infected individuals and $\mathcal{R}_d$ is the reproduction number due to drug users HIV infected individuals.

3.2.3 Stability analysis of disease-free equilibrium point. The jacobian matrix of the system of equations (3.1) to (3.6) evaluated at E_0 is given by

$$J(E_0) = \begin{pmatrix} -\mu & 0 & -\beta S_0 & 0 & -\beta\eta_2 S_0 & 0 \\ 0 & -(\mu+\delta_1) & 0 & -\beta\eta_1 U_0 & 0 & -\beta\eta_3 U_0 \\ 0 & 0 & \beta S_0 - (\mu+\rho_1) & 0 & \beta\eta_2 S_0 & 0 \\ 0 & 0 & 0 & \beta\eta_1 U_0 - (\mu+\rho_2+\delta_1) & 0 & \beta\eta_3 U_0 \\ 0 & 0 & \rho_1 & 0 & -(\mu+\delta_2) & 0 \\ 0 & 0 & 0 & \rho_2 & 0 & -(\mu+\delta_2) \end{pmatrix}.$$

To determine the stability of disease-free equilibrium point, we use $|J(E_0) - \lambda I| = 0$ to obtain eigenvalues of $J(E_0)$.

$$\begin{vmatrix} -\mu-\lambda & 0 & -\beta S_0 & 0 & -\beta\eta_2 S_0 & 0 \\ 0 & -(\mu+\delta_1)-\lambda & 0 & -\beta\eta_1 U_0 & 0 & -\beta\eta_3 U_0 \\ 0 & 0 & \beta S_0 - (\mu+\rho_1)-\lambda & 0 & \beta\eta_2 S_0 & 0 \\ 0 & 0 & 0 & \beta\eta_1 U_0 - (\mu+\rho_2+\delta_1)-\lambda & 0 & \beta\eta_3 U_0 \\ 0 & 0 & \rho_1 & 0 & -(\mu+\delta_2)-\lambda & 0 \\ 0 & 0 & 0 & \rho_2 & 0 & -(\mu+\delta_2)-\lambda \end{vmatrix} = 0$$

$$\lambda_1 = -\mu < 0 \quad \text{and} \quad \lambda_2 = -(\mu+\delta_1) < 0,$$

and

$$\begin{vmatrix} \beta S_0 - (\mu+\rho_1) - \lambda & 0 & \beta\eta_2 S_0 & 0 \\ 0 & \beta\eta_1 U_0 - (\mu+\rho_2+\delta_1) - \lambda & 0 & \beta\eta_3 U_0 \\ \rho_1 & 0 & -(\mu+\delta_2) - \lambda & 0 \\ 0 & \rho_2 & 0 & -(\mu+\delta_2) - \lambda \end{vmatrix} = 0,$$

which gives

$$\Big(\beta S_0 - (\mu+\rho_1) - \lambda\Big) \begin{vmatrix} \beta\eta_1 U_0 - (\mu+\rho_2+\delta_1) - \lambda & 0 & \beta\eta_3 U_0 \\ 0 & -(\mu+\delta_2) - \lambda & 0 \\ \rho_2 & 0 & -(\mu+\delta_2) - \lambda \end{vmatrix}$$

$$+\beta\eta_2 S_0 \begin{vmatrix} 0 & \beta\eta_1 U_0 - (\mu+\rho_2+\delta_1) - \lambda & \beta\eta_3 U_0 \\ \rho_1 & 0 & 0 \\ 0 & \rho_2 & -(\mu+\delta_2) - \lambda \end{vmatrix} = 0,$$

so that either

$$\Big(\beta S_0 - (\mu+\rho_1) - \lambda\Big)\Big(-(\mu+\delta_2) - \lambda\Big) - \beta\eta_2\rho_1 S_0 = 0, \tag{3.12}$$

or

$$\Big(\beta\eta_1 U_0 - (\mu+\rho_2+\delta_1) - \lambda\Big)\Big(-(\mu+\delta_2) - \lambda\Big) - \beta\eta_3\rho_2 U_0 = 0. \tag{3.13}$$

Equation (3.12) reduces to

$$\lambda^2 + \Big((1-\mathcal{R}_h)(\mu+\rho_1) + \psi_1\Big)\lambda + \Big((1-\mathcal{R}_h)(\mu+\rho_1)(\mu+\delta_2)\Big) = 0. \tag{3.14}$$

$$\text{where} \quad \psi_1 = \frac{\beta\eta_2\rho_1 S_0}{(\mu+\delta_2)} + (\mu+\delta_2).$$

The eigenvalues of (3.14) are given by

$$\lambda_{3,4} = -\left[\frac{(1-\mathcal{R}_h)(\mu+\rho_1)+\psi_1}{2}\right] \pm \sqrt{\left(\frac{(1-\mathcal{R}_h)(\mu+\rho_1)+\psi_1}{2}\right)^2 - (1-\mathcal{R}_h)(\mu+\rho_1)(\mu+\delta_2)}$$

$\lambda_{3,4}$ are negative or have negative real parts when $1-\mathcal{R}_h > 0$ i.e. when $\mathcal{R}_h < 1$.

Equation (3.13) reduces to

$$\lambda^2 + \Big((1-\mathcal{R}_d)(\mu+\rho_2+\delta_1) + \psi_2\Big)\lambda + \Big((1-\mathcal{R}_d)(\mu+\rho_2+\delta_1)(\mu+\delta_2)\Big) = 0 \tag{3.15}$$

$$\text{where} \quad \psi_2 = \frac{\beta\eta_3\rho_2 U_0}{(\mu+\delta_2)} + (\mu+\delta_2).$$

The eigenvalues of (3.15) are given by

$$\lambda_{5,6} = -\left[\frac{(1-\mathcal{R}_d)(\mu+\rho_2+\delta_1)+\psi_2}{2}\right] \pm \sqrt{\left(\frac{(1-\mathcal{R}_d)(\mu+\rho_2+\delta_1)+\psi_2}{2}\right)^2 - (1-\mathcal{R}_d)(\mu+\rho_2+\delta_1)(\mu+\delta_2)}$$

$\lambda_{5,6}$ are negative or have negative real parts when $1-\mathcal{R}_d > 0$ i.e. when $\mathcal{R}_d < 1$.

Since $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6$ are all negative or have negative real parts when $\mathcal{R}_h < 1$ or $\mathcal{R}_d < 1$, taking $\mathcal{R}_0 = \max\{\mathcal{R}_h, \mathcal{R}_d\}$, the disease free equilibrium point is locally asymptotically stable when $\mathcal{R}_0 < 1$.

We summarize the stability results for the disease free equilibrium point in the following theorem:

3.2.4 Theorem. *The disease free equilibrium point is locally asymptotically stable when $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$.*

3.2.5 Endemic equilibrium points. Equating the right-hand side of equations (3.1) to (3.6) to zero, we have

$$\begin{aligned}
\pi\nu - \lambda_N^* S^* - \mu S^* &= 0, && (3.16)\\
\pi(1-\nu) - \lambda_D^* U^* - (\mu+\delta_1)U^* &= 0, && (3.17)\\
\lambda_N^* S^* - (\mu+\rho_1)I^* &= 0, && (3.18)\\
\lambda_D^* U^* - (\mu+\rho_2+\delta_1)I_u^* &= 0, && (3.19)\\
\rho_1 I^* - (\mu+\delta_2)A^* &= 0, && (3.20)\\
\rho_2 I_u^* - (\mu+\delta_2)A_u^* &= 0. && (3.21)
\end{aligned}$$

From equation (3.16) to (3.21), we have

$$\begin{aligned}
S^* &= \frac{\pi\nu}{\lambda_N^*+\mu}, && (3.22)\\
U^* &= \frac{\pi(1-\nu)}{\lambda_D^*+\mu+\delta_1}, && (3.23)\\
I^* &= \frac{\lambda_N^*\pi\nu}{(\mu+\rho_1)(\lambda_N^*+\mu)}, && (3.24)\\
I_u^* &= \frac{\lambda_D^*\pi(1-\nu)}{(\mu+\rho_2+\delta_1)(\lambda_D^*+\mu+\delta_1)}, && (3.25)\\
A^* &= \frac{\rho_1\lambda_N^*\pi\nu}{(\mu+\delta_2)(\mu+\rho_1)(\lambda_N^*+\mu)}, && (3.26)\\
A_u^* &= \frac{\rho_2\lambda_D^*\pi(1-\nu)}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)(\lambda_D^*+\mu+\delta_1)}. && (3.27)
\end{aligned}$$

Therefore, the endemic equilibria are given by

$$E^* = (S^*, U^*, I^*, I_u^*, A^*, A_u^*),$$

where $\lambda_N^* = \beta(I^* + \eta_2 A^*) = \beta\pi\nu\lambda_N^*\left(\frac{\mu+\delta_2+\eta_2\rho_1}{(\mu+\delta_2)(\mu+\rho_1)(\lambda_N^*+\mu)}\right)$,

that is

$$\lambda_N^*\left(1 - \frac{\beta\pi\nu(\mu+\delta_2+\eta_2\rho_1)}{(\mu+\delta_2)(\mu+\rho_1)(\lambda_N^*+\mu)}\right) = 0. \quad (3.28)$$

From equation (3.28) we have that either $\lambda_N^* = 0$ or

$$\left(1 - \frac{\beta\pi\nu(\mu+\delta_2+\eta_2\rho_1)}{(\mu+\delta_2)(\mu+\rho_1)(\lambda_N^*+\mu)}\right) = 0, \quad \text{which reduces to} \quad \lambda_N^* = \mu(R_h - 1).$$

This means $\lambda_N^* = 0$ or $\lambda_N^* = \mu(\mathcal{R}_h - 1)$ where $\lambda_N^* > 0$ when $\mathcal{R}_h > 1$.

Similarly, we have

$$\lambda_D^* = \beta(\eta_1 I_u^* + \eta_3 A_u^*) = \beta\pi(1-\nu)\lambda_D^*\left(\frac{\eta_1(\mu+\delta_2)+\eta_3\rho_2}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)(\lambda_D^*+\mu+\delta_1)}\right)$$

$$\lambda_D^*\left(1 - \frac{\beta\pi(1-\nu)(\eta_1(\mu+\delta_2)+\eta_3\rho_2)}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)(\lambda_D^*+\mu+\delta_1)}\right) = 0,$$

so that either $\lambda_D^* = 0$ or

$$1 - \frac{\beta\pi(1-\nu)(\eta_1(\mu+\delta_2)+\eta_3\rho_2)}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)(\lambda_D^*+\mu+\delta_1)} = 0 \quad \text{which reduces to} \quad \lambda_D^* = (\mu+\delta_1)(\mathcal{R}_d - 1)$$

This means either $\lambda_D^* = 0$ or $\lambda_D^* = (\mu+\delta_1)(\mathcal{R}_d - 1)$ where $\lambda_D^* > 0$ when $\mathcal{R}_d > 1$.

The case $\lambda_N^* = \lambda_D^* = 0$ corresponds to the disease-free equilibrium point.

From the existence of endemic equilibrium points we have the following cases:

Case 1: $\lambda_N^* = 0$, $\lambda_D^* = (\mu+\delta_1)(\mathcal{R}_d - 1)$ and $\nu \neq 0 \quad (0 < \nu < 1)$ we have

$$E_D = \left(\frac{\pi\nu}{\mu}, \frac{\pi(1-\nu)}{(\mu+\delta_1)\mathcal{R}_d}, 0, \frac{(\mathcal{R}_d-1)\pi(1-\nu)}{(\mu+\rho_2+\delta_1)\mathcal{R}_d}, 0, \frac{\rho_2(\mathcal{R}_d-1)\pi(1-\nu)}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)\mathcal{R}_d}\right).$$

E_D is an equilibrium point where non-drug users do not contribute to HIV infection but drug users do.

Case 2: $\lambda_N^* = 0$, $\lambda_D^* = (\mu+\delta_1)(\mathcal{R}_d - 1)$ and $\nu = 0$, we have

$$E_{D\text{all}} = \left(0, \frac{\pi}{(\mu+\delta_1)\mathcal{R}_d}, 0, \frac{(\mathcal{R}_d-1)\pi}{(\mu+\rho_2+\delta_1)\mathcal{R}_d}, 0, \frac{\rho_2(\mathcal{R}_d-1)\pi}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)\mathcal{R}_d}\right),$$

an equilibrium point where every individual is on drug and contributing to HIV infection.

Case 3: $\lambda_N^* = \mu(\mathcal{R}_h - 1)$, $\lambda_D^* = 0$ and $\nu \neq 1 \quad (0 < \nu < 1)$, we have

$$E_N = \left(\frac{\pi\nu}{\mu\mathcal{R}_h}, \frac{\pi(1-\nu)}{\mu+\delta_1}, \frac{(\mathcal{R}_h-1)\pi\nu}{(\mu+\rho_1)\mathcal{R}_h}, 0, \frac{\rho_1(\mathcal{R}_h-1)\pi\nu}{(\mu+\delta_2)(\mu+\rho_1)\mathcal{R}_h}, 0\right).$$

E_N is an equilibrium point where non-drug users contribute to HIV infection but drug users do not.

Case 4: $\lambda_N^* = \mu(\mathcal{R}_h - 1)$, $\lambda_D^* = 0$ and $\nu = 1$, we have

$$E_{N\text{all}} = \left(\frac{\pi}{\mu\mathcal{R}_h}, 0, \frac{(\mathcal{R}_h-1)\pi}{(\mu+\rho_1)\mathcal{R}_h}, 0, \frac{\rho_1(\mathcal{R}_h-1)\pi}{(\mu+\delta_2)(\mu+\rho_1)\mathcal{R}_h}, 0\right),$$

an equilibrium point where everyone is not on drug but contributing to HIV infection.

Case 5: $\lambda_N^* = \mu(\mathcal{R}_h - 1)$, $\lambda_D^* = (\mu+\delta_1)(\mathcal{R}_d - 1)$ and $(0 < \nu < 1)$.

$$E_{ND} = \left(\frac{\pi\nu}{\mu\mathcal{R}_h}, \frac{\pi(1-\nu)}{(\mu+\delta_1)\mathcal{R}_d}, \frac{(\mathcal{R}_h-1)\pi\nu}{(\mu+\rho_1)\mathcal{R}_h}, \frac{(\mathcal{R}_d-1)\pi(1-\nu)}{(\mu+\rho_2+\delta_1)\mathcal{R}_d}, \frac{\rho_1(\mathcal{R}_h-1)\pi\nu}{(\mu+\delta_2)(\mu+\rho_1)\mathcal{R}_h}, \frac{\rho_2(\mathcal{R}_d-1)\pi(1-\nu)}{(\mu+\delta_2)(\mu+\rho_2+\delta_1)\mathcal{R}_d}\right),$$

an equilibrium point where both drug users and non-drug users contribute to HIV infection.

We summarize the existence of the endemic equilibrium points in the following theorem:

3.2.6 Theorem. *The system of equations (3.1) to (3.6) has the following endemic equilibrium points:*

1. E_D *and* E_{Dall} *which exist when* $\mathcal{R}_d > 1$.
2. E_N *and* E_{Nall} *which exist when* $\mathcal{R}_h > 1$.
3. E_{ND} *which exists when* $\mathcal{R}_d > 1$ *and* $\mathcal{R}_h > 1$, *i.e.* $\mathcal{R}_0 > 1$.

3.2.7 Stability analysis of endemic equilibrium points. The jacobian matrix of the system of equations (3.1) to (3.6) at an endemic equilibria E^* is

$$J(E^*) = \begin{pmatrix} -(\lambda_N^* + \mu) & 0 & -\beta S^* & 0 & -\beta S^* \eta_2 & 0 \\ 0 & -(\lambda_D^* + \mu + \delta_1) & 0 & -\beta U^* \eta_1 & 0 & -\beta U^* \eta_3 \\ \lambda_N^* & 0 & \beta S^* - (\mu + \rho_1) & 0 & \beta S^* \eta_2 & 0 \\ 0 & \lambda_D^* & 0 & \beta U^* \eta_1 - (\mu + \rho_2 + \delta_1) & 0 & \beta U^* \eta_3 \\ 0 & 0 & \rho_1 & 0 & -(\mu + \delta_2) & 0 \\ 0 & 0 & 0 & \rho_2 & 0 & -(\mu + \delta_2) \end{pmatrix}.$$

3.2.8 Stability of $E^* = E_D$. We observe that $I^* = A^* = 0$, so the equations for $\dot{I}$ and $\dot{A}$ do not contribute anything to the jacobian matrix of E_D.

The system of equations (3.1) to (3.6) reduces to

$$\begin{aligned} \frac{dS}{dt} &= \pi\nu - \mu S, \\ \frac{dU}{dt} &= \pi(1-\nu) - \lambda_D U - (\mu + \delta_1)U, \\ \frac{dI_u}{dt} &= \lambda_D U - (\mu + \rho_2 + \delta_1) I_u, \\ \frac{dA_u}{dt} &= \rho_2 I_u - (\mu + \delta_2) A_u, \end{aligned}$$

where $\lambda_N^* = 0$ and $\lambda_D^* = (\mu + \delta_1)(\mathcal{R}_d - 1)$. The reduced system represents a society where some individuals are not drug users but abstain from spreading HIV infection while some individuals are both drug addicts and spreading HIV infection at the same time.

The jacobian matrix of the system of equations at E_D is

$$J(E_D) = \begin{pmatrix} -\mu & 0 & 0 & 0 \\ 0 & -(\lambda_D^* + \mu + \delta_1) & -\beta U^* \eta_1 & -\beta U^* \eta_3 \\ 0 & \lambda_D^* & -\psi & \beta U^* \eta_3 \\ 0 & 0 & \rho_2 & -(\mu + \delta_2) \end{pmatrix},$$

where

$$\begin{aligned} -\psi &= \beta\eta_1\frac{\pi(1-\nu)}{(\mu+\delta_1)R_d} - (\mu+\rho_2+\delta_1), \\ &= \frac{\beta\eta_1(\mu+\delta_2)(\mu+\rho_2+\delta_1)}{\beta\Big(\rho_2\eta_3+\eta_1(\mu+\delta_2)\Big)} - (\mu+\rho_2+\delta_1), \\ &= -\frac{\rho_2\eta_3(\mu+\rho_2+\delta_1)}{\rho_2\eta_3+\eta_1(\mu+\delta_2)}. \end{aligned}$$

We use $|J(E_D)-\lambda I|=0$ to obtain eigenvalues of $J(E_D)$.

$$|J(E_D)-\lambda I| = \begin{vmatrix} -\mu-\lambda & 0 & 0 & 0 \\ 0 & -(\lambda_D^*+\mu+\delta_1)-\lambda & -\beta U^*\eta_1 & -\beta U^*\eta_3 \\ 0 & \lambda_D^* & -\psi-\lambda & \beta U^*\eta_3 \\ 0 & 0 & \rho_2 & -(\mu+\delta_2)-\lambda \end{vmatrix} = 0.$$

$$\lambda_1 = -\mu < 0,$$

and we have

$$\begin{vmatrix} -(\lambda_D^*+\mu+\delta_1)-\lambda & -\beta U^*\eta_1 & -\beta U^*\eta_3 \\ \lambda_D^* & -\psi-\lambda & \beta U^*\eta_3 \\ 0 & \rho_2 & -(\mu+\delta_2)-\lambda \end{vmatrix} = 0,$$

whose characteristic equation is given by

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \quad \text{where}$$

$$\begin{aligned} a_2 &= (\mu+\delta_2)+\lambda_D^*+(\mu+\delta_1)+\psi, \\ a_1 &= (\mu+\delta_2)\lambda_D^*+\psi(\mu+\delta_1)+(\mu+\delta_1)(\mu+\delta_2)+(\mu+\rho_2+\delta_1)\lambda_D^*, \\ a_0 &= (\mu+\delta_2)(\mu+\rho_2+\delta_1)\lambda_D^*. \end{aligned}$$

We use the Routh-Hurwitz stability criterion for a third order polynomial to ascertain that all the eigenvalues of $J(E_D)$ are either negative or have negative real parts. The following conditions should hold:

$$a_2 > 0, \quad a_0 > 0 \quad \text{and} \quad a_2a_1 - a_0 > 0.$$

Clearly, $a_2 > 0$ when $\mathcal{R}_d > 1$ and $a_0 > 0$ when $\mathcal{R}_d > 1$.

Now

$a_2a_1 - a_0 = \Big((\mu+\delta_2)+\lambda_D^*+(\mu+\delta_1)+\psi\Big)\Big((\mu+\delta_2)\lambda_D^*+\psi(\mu+\delta_1)+(\mu+\delta_1)(\mu+\delta_2)+(\mu+\rho_2+\delta_1)\lambda_D^*\Big) - (\mu+\delta_2)(\mu+\rho_2+\delta_1)\lambda_D^*,$

$= (\mu+\delta_2)^2\lambda_D^*+\psi(\mu+\delta_1)(\mu+\delta_2)+(\mu+\delta_1)(\mu+\delta_2)^2+(\mu+\delta_2)(\mu+\rho_2+\delta_1)\lambda_D^*+(\mu+\delta_2){\lambda_D^*}^2$

$+\psi\lambda_D^*(\mu+\delta_1)+\lambda_D^*(\mu+\delta_1)(\mu+\delta_2)+(\mu+\rho_2+\delta_1){\lambda_D^*}^2+(\mu+\delta_1)(\mu+\delta_2)\lambda_D^*+\psi(\mu+\delta_1)^2$
$+(\mu+\delta_1)^2(\mu+\delta_2)+(\mu+\delta_1)(\mu+\rho_2+\delta_1)\lambda_D^*+(\mu+\delta_2)\psi\lambda_D^*+\psi^2(\mu+\delta_1)+\psi(\mu+\delta_1)(\mu+\delta_2)$
$+(\mu+\rho_2+\delta_1)\psi\lambda_D^*-(\mu+\delta_2)(\mu+\rho_2+\delta_1)\lambda_D^*>0$, when $\lambda_D^*>0$ i.e. when $\mathcal{R}_d>1$.

All the Routh-Hurwitz criterion conditions are satisfied when $\mathcal{R}_d>1$. E_D is asymptotically stable when $\mathcal{R}_d>1$.

3.2.9 Stability of $E^*=E_{D\text{all}}$. We observe that $S^*=I^*=A^*=0$, so the equations for $\dot{S}$, $\dot{I}$ and $\dot{A}$ do not contribute to the jacobian of $E_{D\text{all}}$ when $\nu=0$.

The system of equations (3.1) to (3.6) reduces to

$$\frac{dU}{dt} = \pi-\lambda_D U-(\mu+\delta_1)U, \tag{3.29}$$
$$\frac{dI_u}{dt} = \lambda_D U-(\mu+\rho_2+\delta_1)I_u, \tag{3.30}$$
$$\frac{dA_u}{dt} = \rho_2 I_u-(\mu+\delta_2)A_u, \tag{3.31}$$

where $\lambda_N^*=0$ and $\lambda_D^*=(\mu+\delta_1)(\mathcal{R}_d-1)$. The system represents a society where every individual is a drug addict and spreading HIV infection.

The jacobian matrix of the system of equations (3.29) to (3.31) at $E_{D\text{all}}$ is

$$J(E_{D\text{all}})=\begin{pmatrix}-(\lambda_D^*+\mu+\delta_1) & -\beta U^*\eta_1 & -\beta U^*\eta_3\\ \lambda_D^* & -\psi & \beta U^*\eta_3\\ 0 & \rho_2 & -(\mu+\delta_2)\end{pmatrix}.$$

We will use $|J(E_{D\text{all}})-\lambda I|=0$ to obtain eigenvalues of $J(E_{D\text{all}})$.

$$|J(E_{D\text{all}})-\lambda I|=\begin{vmatrix}-(\lambda_D^*+\mu+\delta_1)-\lambda & -\beta U^*\eta_1 & -\beta U^*\eta_3\\ \lambda_D^* & -\psi-\lambda & \beta U^*\eta_3\\ 0 & \rho_2 & -(\mu+\delta_2)-\lambda\end{vmatrix}=0$$

has a characteristic equation given by

$$\lambda^3+b^2\lambda^2+b_1\lambda b_0=0, \quad \text{where}$$

$$\begin{aligned} b_2 &= (\mu+\delta_2)+\lambda_D^*+(\mu+\delta_1)+\psi,\\ b_1 &= (\mu+\delta_2)\lambda_D^*+\psi(\mu+\delta_1)+(\mu+\delta_1)(\mu+\delta_2)+(\mu+\rho_2+\delta_1)\lambda_D^*,\\ b_0 &= (\mu+\delta_2)(\mu+\rho_2+\delta_1)\lambda_D^*. \end{aligned}$$

We also use the Routh-Hurwitz stability criterion for a third order polynomial to ascertain that all the eigenvalues of $J(E_{D\text{all}})$ are either negative or have negative real parts by showing that the following conditions hold:

$$b_2>0, \quad b_0>0, \quad \text{and} \quad b_2b_1-b_0>0.$$

Clearly $b_2 > 0$ when $\mathcal{R}_d > 1$ and $b_0 > 0$ when $\mathcal{R}_d > 1$.

$b_2 b_1 - b_0 = \Big((\mu+\delta_2) + \lambda_D^* + (\mu+\delta_1) + \psi\Big)\Big((\mu+\delta_2)\lambda_D^* + \psi(\mu+\delta_1) + (\mu+\delta_1)(\mu+\delta_2) + (\mu+\rho_2+\delta_1)\lambda_D^*\Big) - (\mu+\delta_2)(\mu+\rho_2+\delta_1)\lambda_D^*,$

$= (\mu+\delta_2)^2\lambda_D^* + \psi(\mu+\delta_1)(\mu+\delta_2) + (\mu+\delta_1)(\mu+\delta_2)^2 + (\mu+\delta_2)(\mu+\rho_2+\delta_1)\lambda_D^* + (\mu+\delta_2){\lambda_D^*}^2$
$+\psi\lambda_D^*(\mu+\delta_1) + \lambda_D^*(\mu+\delta_1)(\mu+\delta_2) + (\mu+\rho_2+\delta_1){\lambda_D^*}^2 + (\mu+\delta_1)(\mu+\delta_2)\lambda_D^* + \psi(\mu+\delta_1)^2$
$+(\mu+\delta_1)^2(\mu+\delta_2) + (\mu+\delta_1)(\mu+\rho_2+\delta_1)\lambda_D^* + (\mu+\delta_2)\psi\lambda_D^* + \psi^2(\mu+\delta_1) + \psi(\mu+\delta_1)(\mu+\delta_2)$
$+(\mu+\rho_2+\delta_1)\psi\lambda_D^* - (\mu+\delta_2)(\mu+\rho_2+\delta_1)\lambda_D^* > 0$ when $\lambda_D^* > 0$ i.e. $\mathcal{R}_d > 1$.

The Routh-Hurwitz criterion conditions are satisfied when $\mathcal{R}_d > 1$, so $E_{D\text{all}}$ is asymptotically stable when $\mathcal{R}_d > 1$.

3.2.10 Stability of $E^* = E_N$. $I_u^* = A_u^* = 0$ and so the equations for $\dot{I}_u$ and $\dot{A}_u$ do not contribute to the jacobian matrix of E_N.

The system of equations (3.1) to (3.6) reduces to

$$\frac{dS}{dt} = \pi\nu - \lambda_N S - \mu S, \tag{3.32}$$
$$\frac{dU}{dt} = \pi(1-\nu) - (\mu+\delta_1)U, \tag{3.33}$$
$$\frac{dI}{dt} = \lambda_N S - (\mu+\rho_1)I, \tag{3.34}$$
$$\frac{dA}{dt} = \rho_1 I - (\mu+\delta_2)A, \tag{3.35}$$

where $\lambda_N^* = \mu(\mathcal{R}_h - 1)$ and $\lambda_D^* = 0$. The system represents a society with the drug users not spreading the HIV infection and non-drug users spreading the infection.

The jacobian matrix of the system of equations (3.32) to (3.35) at E_N is

$$J(E_N) = \begin{pmatrix} -(\lambda_N^* + \mu) & 0 & -\beta S^* & -\beta\eta_2 S^* \\ 0 & -(\mu+\delta_1) & 0 & 0 \\ \lambda_N^* & 0 & -\varphi & \beta\eta_2 S^* \\ 0 & 0 & \rho_1 & -(\mu+\delta_2) \end{pmatrix}, \tag{3.36}$$

where

$$\varphi = \beta\frac{\pi\nu}{\mu R_h} - (\mu+\rho_1), \tag{3.37}$$
$$= \frac{(\mu+\delta_2)(\mu+\rho_1)}{\rho_1\eta_2 + \mu + \delta_2} - (\mu+\rho_1), \tag{3.38}$$
$$= -\frac{\rho_1\eta_2(\mu+\rho_1)}{\rho_1\eta_2 + \mu + \delta_2}. \tag{3.39}$$

We use $|J(E_N) - \lambda I| = 0$ to obtain eigenvalues of $J(E_N)$.

$$|J(E_N) - \lambda I| = \begin{vmatrix} -(\lambda_N^* + \mu) - \lambda & 0 & -\beta S^* & -\beta\eta_2 S^* \\ 0 & -(\mu+\delta_1) - \lambda & 0 & 0 \\ \lambda_N^* & 0 & -\varphi - \lambda & \beta\eta_2 S^* \\ 0 & 0 & \rho_1 & -(\mu+\delta_2) - \lambda \end{vmatrix} = 0.$$

$$\lambda_1 = -(\mu + \delta_1) < 0,$$

and the rest of the eigenvalues are roots of the cubic equation

$$\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0, \quad \text{where}$$

$$\begin{aligned} c_2 &= (\mu+\delta_2) + \lambda_N^* + \mu + \varphi, \\ c_1 &= (\mu+\delta_2)\lambda_N^* + \varphi\mu + \mu(\mu+\delta_2) + (\mu+\rho_1)\lambda_N^*, \\ c_0 &= (\mu+\rho_1)(\mu+\delta_2)\lambda_N^*. \end{aligned}$$

$$c_2 > 0 \quad \text{and} \quad c_0 > 0 \quad \text{when} \quad \mathcal{R}_h > 1.$$

$$c_2c_1 - c_0 = \Big((\mu+\delta_2) + \lambda_N^* + \mu + \varphi\Big)\Big((\mu+\delta_2)\lambda_N^* + \varphi\mu + \mu(\mu+\delta_2) + (\mu+\rho_1)\lambda_N^*\Big) - (\mu+\rho_1)(\mu+\delta_2)\lambda_N^*$$

$$= (\mu+\delta_2)^2\lambda_N^* + \varphi\mu(\mu+\delta_2) + \mu(\mu+\delta_2)^2 + (\mu+\rho_1)(\mu+\delta_2)\lambda_N^* + (\mu+\delta_2){\lambda_N^*}^2 + \varphi\mu\lambda_N^*$$
$$+\mu\lambda_N^*(\mu+\delta_2) + (\mu+\rho_1){\lambda_N^*}^2 + (\mu+\delta_2)\mu\lambda_N^* + \varphi\mu^2 + \mu^2(\mu+\delta_2) + (\mu+\rho_1)\mu\lambda_N^* + (\mu+\delta_2)\varphi\lambda_N^*$$
$$+\varphi^2\mu + \mu\varphi(\mu+\delta_2) + (\mu+\rho_1)\varphi\lambda_N^* - (\mu+\rho_1)(\mu+\delta_2)\lambda_N^* > 0 \text{ when } \lambda_N^* > 0 \text{ i.e when } \mathcal{R}_h > 1$$

The eigenvalues for the jacobian matrix 3.36 are negative or have negative real parts when $\mathcal{R}_h > 1$. E_N is asymptotically stable when $R_h > 1$.

3.2.11 Stability of $E^* = E_{N\text{all}}$. In this case $U^* = I_u^* = A_u^* = 0$, and when $\nu = 1$ the system of equations (3.1) to (3.6) reduces to

$$\frac{dS}{dt} = \pi - \lambda_N S - \mu S, \tag{3.40}$$
$$\frac{dI}{dt} = \lambda_N S - (\mu+\rho_1)I, \tag{3.41}$$
$$\frac{dA}{dt} = \rho_1 I - (\mu+\delta_2)A, \tag{3.42}$$

where $\lambda_N^* = \mu(\mathcal{R}_h - 1)$ and $\lambda_D^* = 0$. The system represents a society with non-drug users only and spreading HIV infection.

The jacobian matrix of the system of equations (3.40) to (3.42) at $E_{N\text{all}}$ is

$$J(E_{N\text{all}}) = \begin{pmatrix} -(\lambda_N^* + \mu) & -\beta S^* & -\beta\eta_2 S^* \\ \lambda_N^* & -\varphi & \beta\eta_2 S^* \\ 0 & \rho_1 & -(\mu+\delta_2) \end{pmatrix}. \tag{3.43}$$

To obtain the eigenvalues of $J(E_{N\text{all}})$, we use $|J(E_{N\text{all}}) - \lambda I| = 0$.

$$|J(E_{N\text{all}}) - \lambda I| = \begin{vmatrix} -(\lambda_N^* + \mu) - \lambda & -\beta S^* & -\beta\eta_2 S^* \\ \lambda_N^* & -\varphi - \lambda & \beta\eta_2 S^* \\ 0 & \rho_1 & -(\mu + \delta_2) - \lambda \end{vmatrix} = 0,$$

has a characteristic equation given by

$$\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0 = 0, \quad \text{where}$$

$$\begin{aligned} d_2 &= (\mu + \delta_2) + \lambda_N^* + \mu + \varphi > 0 \quad \text{when} \quad \mathcal{R}_h > 1, \\ d_1 &= (\mu + \delta_2)\lambda_N^* + \varphi\mu + \mu(\mu + \delta_2) + (\mu + \rho_1)\lambda_N^*, \\ d_0 &= (\mu + \rho_1)(\mu + \delta_2)\lambda_N^* > 0 \quad \text{when} \quad \mathcal{R}_h > 1. \end{aligned}$$

The results for $d_2d_1 - d_0 > 0$ are similar to the analysis for E_N and so the Routh-Hurwitz conditions are satisfied when $\mathcal{R}_h > 1$. $E_{N\text{all}}$ is asymptotically stable when $\mathcal{R}_h > 1$.

3.2.12 Stability of $E^* = E_{ND}$. Rearranging the system of equations (3.1) to (3.6) we have:

$$\frac{dS}{dt} = \pi\nu - \lambda_N S - \mu S, \tag{3.44}$$
$$\frac{dI}{dt} = \lambda_N S - (\mu + \rho_1)I, \tag{3.45}$$
$$\frac{dA}{dt} = \rho_1 I - (\mu + \delta_2)A, \tag{3.46}$$
$$\frac{dU}{dt} = \pi(1 - \nu) - \lambda_D U - (\mu + \delta_1)U, \tag{3.47}$$
$$\frac{dI_u}{dt} = \lambda_D U - (\mu + \rho_2 + \delta_1)I_u, \tag{3.48}$$
$$\frac{dA_u}{dt} = \rho_2 I_u - (\mu + \delta_2)A_u, \tag{3.49}$$

where $\lambda_N^* = \mu(\mathcal{R}_h - 1)$ and $\lambda_D^* = (\mu + \delta_1)(\mathcal{R}_d - 1)$.

The jacobian matrix of the system of equations (3.44) to (3.49) at E_{ND} is

$$J(E_{ND}) = \begin{pmatrix} -(\lambda_N^* + \mu) & -\beta S^* & -\beta\eta_2 S^* & 0 & 0 & 0 \\ \lambda_N^* & -\varphi & \beta\eta_2 S^* & 0 & 0 & 0 \\ 0 & \rho_1 & -(\mu + \delta_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\lambda_D^* + \mu + \delta_1) & -\beta U^*\eta_1 & -\beta U^*\eta_3 \\ 0 & 0 & 0 & \lambda_D^* & -\psi & \beta U^*\eta_3 \\ 0 & 0 & 0 & 0 & \rho_2 & -(\mu + \delta_2) \end{pmatrix}.$$

We observe that $J(E_{ND})$ can be written as

$$J(E_{ND}) = \begin{pmatrix} J(E_{N\text{all}}) & 0 \\ 0 & J(E_{D\text{all}}) \end{pmatrix}.$$

The eigenvalues of $J(E_{ND})$ are eigenvalues of the block matrices $J(E_{D\text{all}})$ and $J(E_{N\text{all}})$.

Since the eigenvalues of $J(E_{N\text{all}})$ are negative or have negative real parts when $\mathcal{R}_h > 1$ and those for $J(E_{D\text{all}})$ are negative or have negative real parts when $\mathcal{R}_d > 1$, it follows that all the eigenvalues of $J(E_{ND})$ are negative or have negative real parts when $\mathcal{R}_h > 1$ and $\mathcal{R}_d > 1$, that is when $\mathcal{R}_0 > 1$.

We summarize the stability of the endemic points in the following theorem:

3.2.13 Theorem. *1. E_N and $E_{N\text{all}}$ are locally asymptotically stable when $\mathcal{R}_h > 1$.*

2. E_D and $E_{D\text{all}}$ are locally asymptotically stable when $\mathcal{R}_d > 1$.

3. E_{ND} is locally asymptotically stable when $\mathcal{R}_h > 1$ and $\mathcal{R}_d > 1$.

Remarks:

Our model revealed the following scenarios regarding the effects of drug abuse and HIV infection in an endemic section:

1. A scenario where everyone is not using drug and spreading HIV infection.
2. A scenario where everyone is a drug user and spreading HIV infection.
3. A scenario where there are drug users and non-drug users and all are spreading HIV infection.
4. A scenario with non-drug users and drug users but non-drug users abstain completely from spreading HIV infection whilst drug users spread HIV infection.
5. A scenario with non-drug users and drug users but drug users are not spreading HIV infection.

We shall explore the impact of these scenario on HIV infection progression using numerical simulations.

4. Numerical Simulations

All parameter values used in the simulations are given in table 4.1. Using the parameter values in table 4.1, the initial conditions for the susceptible non-drug users S and drug users U were obtained using the fact that when there are no HIV infected individuals, $S_0 = \frac{\pi\nu}{\mu}$ and $U_0 = \frac{\pi(1-\nu)}{\mu+\delta_1}$. The resultant expressions for S_0 and U_0 are given by

$$S_0 = 4 \times 10^5 \nu \quad \text{and} \quad U_0 = 1.7241 \times 10^5 (1-\nu), \quad 0 \leq \nu \leq 1.$$

Table 4.2 shows the initial conditions for different values of ν.

Table 4.1: Parameter values and their sources

Parameter	Value	Source
π	10000	Hove-Musekwa and Nyabadza (2009)
μ	0.025	Kalula and Nyabadza (2012)
ρ_1	0.125	Baryarama et al. (2005)
ρ_2	0.143	estimate
δ_1	0.033	Kalula and Nyabadza (2012)
δ_2	0.5	Baryarama et al. (2005)
β	0.00001	estimate
η_1	1.1	estimate
η_2	0.02	estimate
η_3	0.04	estimate

Table 4.2: Initial conditions for different values of ν.

ν	S_0	U_0	I_0	I_{u0}	A_0	A_{u0}
0.0	0	1.7241×10^5	0	1	0	0
0.2	80000	1.3793×10^5	0	1	0	0
0.5	200000	8.6207×10^4	1	1	0	0
0.8	320000	3.4483×10^4	1	0	0	0
1.0	400000	0	1	0	0	0

4.1 Numerical simulations for different values of ν and discussion of results

Figure 4.1 represents a situation where all the susceptible individuals are drug users i.e. $\nu = 0$. This represents the dynamics of a society with the equilibrium point $E_{D\text{all}}$. Figure 4.1 shows that the susceptible drug users decreases while the infected drug users and AIDS drug users increases. Infected drug users settle at higher levels than the AIDS drug users and susceptible drug users. In this case the population has a problem of only drug users that are infected with HIV and have a higher potential to spread HIV infection if there is no intervention. This may enhance the faster progression of infected individuals to AIDS since all infected drug users will ultimately develop AIDS and die.

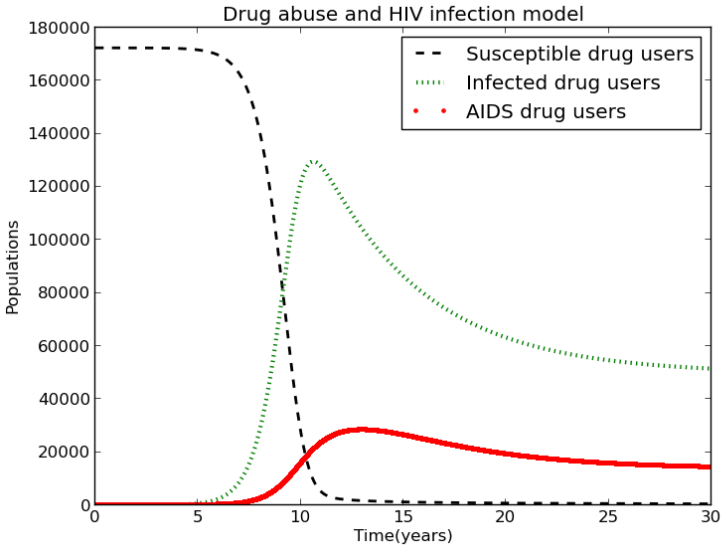

Figure 4.1: The graph of drug abuse and HIV infection at $\nu = 0$

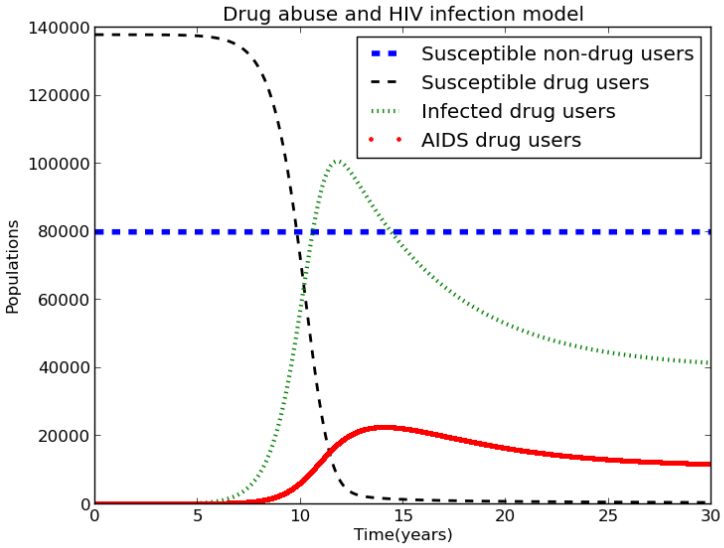

Figure 4.2: The graph of drug abuse and HIV infection at $\nu = 0.2$

From figure 4.2 we have that at $\nu = 0.2$, infected non-drug users and AIDS non-drug users dynamics are at zero while a population of susceptible non-drug users is not zero but not participating in the HIV infection dynamics. This graph represents the equilibrium point E_D. At the equilibrium point, susceptible drug users decrease. Infected drug users and AIDS drug users increase and infected drug users settle at higher levels than the AIDS drug users and susceptible drug users. The levels of infected non-drug users and AIDS non-drug users in this case is lower than that of 4.1. This suggests that when some individuals are abstaining from transmitting HIV, the level of transmission of HIV reduces.

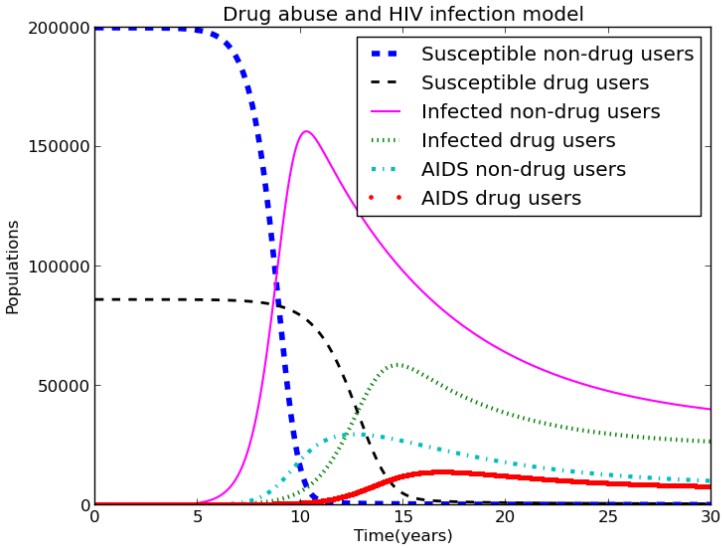

Figure 4.3: The graph of drug abuse and HIV infection at $\nu = 0.5$

From figure 4.3 we have that at $\nu = 0.5$, all drug users and non-drug users dynamics contribute to HIV infection but drug users contributed more than than non-drug users. This is a situation for the equilibrium E_{ND}. At the equilibrium point, the susceptible drug users and non-drug users decreases but susceptible drug users decreases faster than non-drug users. The infected drug users and non-drug users increases but drug users increases faster than non-drug users. AIDS drug users and non-drug users increases but drug users increases faster than non-drug users. Infected non-drug users settle at higher levels than the others. The total number of infected individuals and AIDS individuals is more than those in figures 4.1 and 4.2. Thus if both drug users and non-drug users contribute towards HIV transmission, the level of transmission gets much higher and worsen the HIV epidermic.

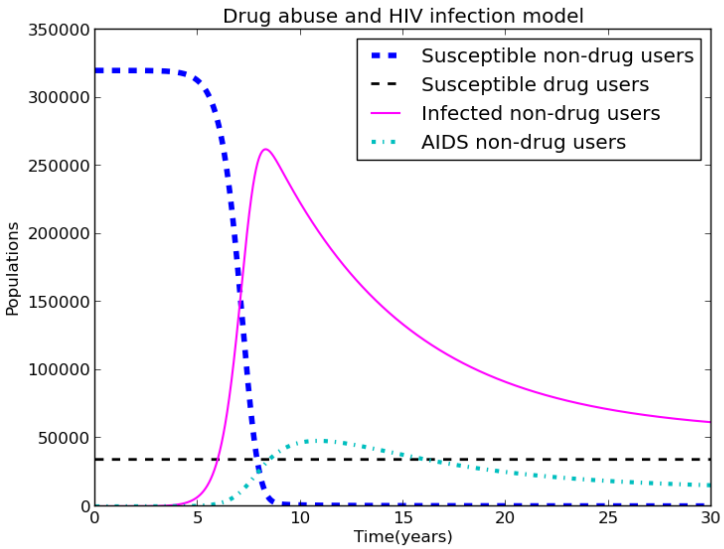

Figure 4.4: The graph of drug abuse and HIV infection at $\nu = 0.8$

From figure 4.4 we have that at $\nu = 0.8$, infected drug users and AIDS drug users dynamics are at zero but the population of susceptible drug users is not zero but not participating in the HIV infection dynamics. This graph represents the equilibrium point E_N. At the equilibrium point, the susceptible drug users are constant while the susceptible non-drug users decrease. Infected non-drug users and AIDS non-drug users increase and infected non-drug users settle at higher levels than the AIDS non-drug users and susceptible non-drug users. The levels of Infected non-drug users and AIDS non-drug users in this case is lower than that of figures 4.3 and 4.5 but higher than that of figure 4.2. This scenario reduces the level of transmission of HIV but the society will still have a problem of drug users.

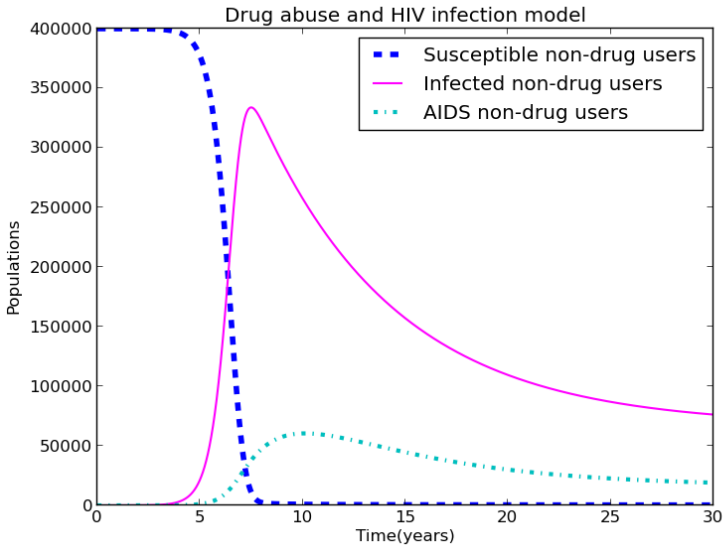

Figure 4.5: The graph of drug abuse and HIV infection at $\nu = 1$

Figure (4.5) shows a situation where everyone is not on drugs but contributing towards HIV infection.This graph represents the equilibrium point $E_{N\text{all}}$. The results in figure 4.5 show that susceptible non-drug users decreases while the infected non-drug users and AIDS non-drug users increase. Infected non-drug users settle at higher levels than the AIDS non-drug users and susceptible non-drug users. The population has a higher number of infected individuals exposing the population to a faster progression towards the AIDS class. However, the rate of population progressing to AIDS may be slower than that of a population with drug users.

4.2 Conclusion

We considered a mathematical model of equations on drug abuse and HIV infection. We denoted the population of susceptible non-drug users by S, the population of susceptible drug users by U, the population of infected non-drug users by I, the population of infected drug users by I_u, the population of AIDS non-drug users by A and the population of AIDS drug users by A_u.

We calculated the threshold parameter $\mathcal{R}_0$ which determined conditions under which the HIV could be transmitted and remained endemic in both drug users and non-drug users population. Thus we showed that when $\mathcal{R}_0 < 1$, only the disease-free equilibrium point exists and it is locally asymptotically stable.

We showed that in a population with both drug users and non-drug users, there are more than one equilibrium points which exist and are locally asymptotically stable when $\mathcal{R}_0 > 1$.

We used numerical simulations to compare the hypothetical endemic scenarios revealed by our analytical results. Our simulations were purely hypothetical since they did not use data for a particular community but showed the qualitative feature that revealed the impact of each of the scenarios on HIV transmission. Our results suggested that the scenarios where some of the individuals abstained from transmitting HIV (figures 4.2 and 4.4) have lower HIV transmission levels and the worst scenario was where both drug users and non-drug users contributed towards HIV transmission. Thus, a society with some individuals abusing drugs is at the worst risk of spreading HIV which in turn creates socio-economic effects if no intervention is implemented in time for both drug use and HIV.

As part of future work to improve the model in this project, the model considered can be restructured to incorporate infection of drug users by non-drug users and infection of non-drug users by drug users. The model can also be refined to incorporate intervention strategies and allow non-drug users to get addicted. In the future work, we wish to find the global stability of the model and equilibrium points. Despite all its limitations, the model provided useful information and insights into the potential impact of drug abuse on the progression of HIV infection.

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