Final Review

1.1 # 33 f)

Construct a truth table for $(p \to q) \to (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

1.4 # 12 f)

Let $Q(x)$ be the statement "$x+1 > 2x$". If the domain consists of all integers what is the true value of $\exists x \neg Q(x)$?

True, for $x=2$ we have $x+1 = 3 \leq 4 = 2x$. So $Q(2)$ is false and hence $\neg Q(2)$ is true.

1.5 #31. b)

Express the negation of $\forall x \exists y P(x,y) \vee \forall x \exists y Q(x,y)$

$\neg(\forall x \exists y P(x,y) \vee \forall x \exists y Q(x,y))$

$\equiv \exists x \forall y \neg P(x,y) \wedge \exists x \forall y \neg Q(x,y)$

2.1. #11

Determine whether each of the following is true or false

a) $0 \in \emptyset$ False

b) $\emptyset \in \{0\}$ False

c) $\{0\} \subseteq \emptyset$ False

d) $\emptyset \subseteq \{0\}$ True

e) $\{0\} \in \{0\}$ ~~True~~ False

g) $\{0\} \subseteq \{0\}$ True

h) $\{0\} \in \{\{0\}\}$

True

2.3 #12 d)

Determine whether $f: \mathbb{Z} \to \mathbb{Z}$ is one-to-one if $f(n) = \lceil n/2 \rceil$.

Not One-to-one

The function $f: \mathbb{Z} \to \mathbb{Z}$ is not one-to-one because

$$f(1) = 1 = f(2).$$

$$\left\lceil \tfrac{1}{2} \right\rceil \qquad \left\lceil \tfrac{2}{2} \right\rceil$$

f is onto. ~~Take~~ take arbitrary $b \in \mathbb{Z}$

Want $n \in \mathbb{Z}$ s.t. $f(n) = b$

Take $n = 2b$, then $f(2b) = \left\lceil \frac{2b}{2} \right\rceil = b$ ☒

3.2 # 10

Show x^3 is $O(x^4)$ but x^4 is Not $O(x^3)$

$$|x^3| \leq 1 \cdot |x^4| \quad \text{for all } x > 1$$

This show x^3 is $O(x^4)$ with witness $C = 1$ and $k = 1$.

Assume for a contradiction that x^4 is $O(x^3)$, then there exists constants C and k such that

$$|x^4| \leq C \cdot |x^3| \quad \text{for all } x > k$$

this would imply

$$|x| \leq C \quad \text{for all } x > k$$

but here we get a contradiction for $x > \max\{k, C\}$.

5.1 #14

Prove that for every positive integer n

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2.$$

Base case: $n=1$

LHS: $\sum_{k=1}^{1} k2^k = 2$

RHS: $(1-1)2^2 + 2 = 2$

Inductive step: Assume that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$

for some $n \geq 1$.

$$\sum_{k=1}^{n+1} k2^k = \sum_{k=1}^{n} k2^k + (n+1)2^{n+1}$$

$$= (n-1)2^{n+1} + 2 + (n+1)2^{n+1}$$

$$= 2n \cdot 2^{n+1} + 2$$

$$= n2^{n+2} + 2 \leftarrow \text{RHS for } n+1$$

$$(n+1-1)2^{n+1+1} + 2$$

Therefore

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$$

for all n by induction.