SC/MATH 1019B - SET OPERATIONS AND INTERVALS
IN CLASS ACTIVITY SEPTEMBER 25, 2018

The purpose of this activity is to practice working with set operations. We will use use certain subsets of $\mathbb{R}$ known as *intervals* as our main focus. So, for this activity $\mathbb{R}$ will be our universal set. Let $a, b \in \mathbb{R}$ be any two real numbers. The *closed interval* $[a, b]$ is defined by

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

The *open interval* (a, b) is defined by

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

There are two *half-open intervals* (or alternatively *half-closed intervals*) $(a, b]$ and $[a, b)$ defined by

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

and

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

respectively. The term *interval* can mean any of these four types. We will specify which type of interval explicitly when necessary. We also have the intervals

$$\begin{aligned}
(-\infty, b] &= \{x \in \mathbb{R} : x \leq b\} \\
(-\infty, b) &= \{x \in \mathbb{R} : x < b\} \\
[a, \infty) &= \{x \in \mathbb{R} : a \leq x\} \\
(a, \infty) &= \{x \in \mathbb{R} : a < x\} \\
(-\infty, \infty) &= \mathbb{R}.
\end{aligned}$$

Question 1. *Can an interval ever be the empty set? Can an interval ever be a finite set? Try to find all possible cases where an interval is empty or finite.*

The intervals $(a, a), (a, a]$, and $[a, a)$ are empty for any $a \in \mathbb{R}$ while the interval $[a, a] = \{a\}$ is finite. Also whenever $a, b \in \mathbb{R}$ with $a > b$ all the intervals $(a, b), (a, b], [a, b), and [a, b]$ are empty.

We can perform set operations on intervals to get new subsets of $\mathbb{R}$. When performing set operations on intervals we may or may not get another interval. However, we can often express the result in terms of intervals.

Example 1. We can find that

$$\begin{aligned}
[1, 3) \cup (0, 2) &= (0, 3) \\
(-\infty, 7) \cup (-4, \infty) &= \mathbb{R} \\
\Big((-6, -4) \cup (4, 6)\Big) \cap [-5, 5] &= [-5, -4) \cup (4, 5]
\end{aligned}$$

for example.

Question 2. *What can you say about the complement of a closed interval? How about the complement of an open interval? Try describing these complements as unions of intervals.*

The complements are $\overline{[a,b]} = (-\infty, a) \cup (b, \infty)$ and $\overline{(a,b)} = (\infty, a] \cup [b, \infty)$.

For any $n \in \mathbb{Z}$ define the following intervals

$$\begin{aligned} I_n &= [n, n+1] \\ J_n &= (n, n+1) \\ K_n &= (n, n+1] \\ L_n &= [n, n+1). \end{aligned}$$

Question 3. *Consider the unions*

$$\bigcup_{n\in\mathbb{Z}} I_n \qquad \bigcup_{n\in\mathbb{Z}} J_n \qquad \bigcup_{n\in\mathbb{Z}} K_n \qquad \bigcup_{n\in\mathbb{Z}} L_n.$$

What can you say about these unions? Are any of them equal?

$$\bigcup_{n\in\mathbb{Z}} I_n = \mathbb{R} \qquad \bigcup_{n\in\mathbb{Z}} J_n = \mathbb{R} \setminus \mathbb{Z} = \overline{\mathbb{Z}} \qquad \bigcup_{n\in\mathbb{Z}} K_n = \mathbb{R} \qquad \bigcup_{n\in\mathbb{Z}} L_n = \mathbb{R}$$

For each $n \in \mathbb{Z}^+$ define

$$A_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$$

and also

$$B_n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right).$$

Question 4. *Consider the unions and intersections*

$$\bigcup_{n\in\mathbb{Z}^+} A_n \qquad \bigcap_{n\in\mathbb{Z}^+} A_n \qquad \bigcup_{n\in\mathbb{Z}^+} B_n \qquad \bigcap_{n\in\mathbb{Z}^+} B_n.$$

What can you say about these subsets of $\mathbb{R}$? Can you describe them as an interval? In which cases is the interval open? In which cases is it closed?

$$\bigcup_{n\in\mathbb{Z}^+} A_n = (0,1) \qquad \bigcap_{n\in\mathbb{Z}^+} A_n = \emptyset \qquad \bigcup_{n\in\mathbb{Z}^+} B_n = (-1,2) \qquad \bigcap_{n\in\mathbb{Z}^+} B_n = [0,1].$$