Chapter 9 Relations

Let A and B be two sets. A relation (binary relation) from A to B is a subset of $A \times B$.

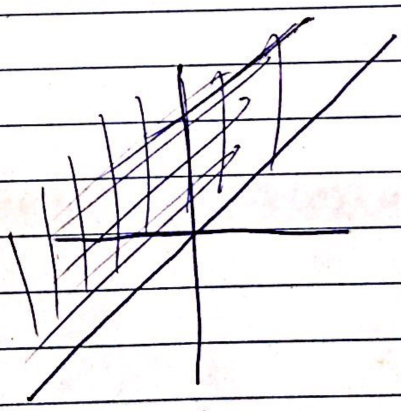
A relation on a set A is a subset of $A \times A$

$R_1 = \{(a,b) : a < b\} \subseteq \mathbb{R} \times \mathbb{R}$

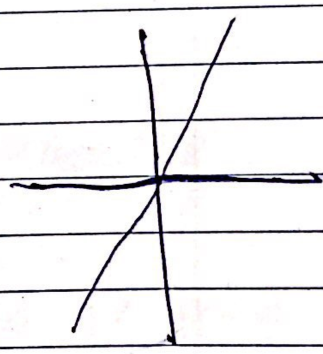
$R_2 = \{(a,b) : a \leq b\} \subseteq \mathbb{R} \times \mathbb{R}$

$R_3 = \{(a,b) : b = 2a\} \subseteq \mathbb{R} \times \mathbb{R}$

R_1



R_2



R_3

Let R be a relation of a set A.

R is symmetric if $(a,b) \in R$ implies $(b,a) \in R$

R is reflexive if $(a,a) \in R$ for all $a \in A$

R is transitive if $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$.

$A = \{a,b,c,d\}$

$R_1 = \{(a,a), (b,b), (c,c), (d,d)\}$

$R_2 = \{(a,b), (b,c), (c,d)\}$

$R_3 = \{(a,b), (b,c), (c,d), (d,a)\}$

$R_4 = \{(b,d)\}$

Which are symmetric? reflexive? transitive?

An equivalence relation is a relation which is symmetric, reflexive, and transitive.

Which of the above are equivalence relations?

$R_1 = \{(x,y) : xy \geq 0\} \subseteq \mathbb{R} \times \mathbb{R}$

$R_2 = \{(x,y) : x-y \in \mathbb{Z}\} \subseteq \mathbb{R} \times \mathbb{R}$

Is R_1 an equivalence relation? How about R_2?

R_1 is reflexive since $(x,x) \in R_1$ $\forall x \in \mathbb{R}$ because $x^2 \geq 0$ $\forall$ $x \in \mathbb{R}$.

R_1 is symmetric since $(x,y) \in R_1$ means $xy \geq 0$ so also $yx \geq 0$ and $(y,x) \in R_1$.

However R_1 is NOT transitive.

$(-1,0) \in R_1$, $(0,1) \in R_1$

But $(-1,1) \notin R_1$. So R_1 is not an equivalence relation

R_2 is an equivalence relation. It's reflexive because $x-x=0 \in \mathbb{Z}$ $\forall x \in \mathbb{R}$ so $(x,x) \in R_2$.
It's ~~symmetric~~ because if we have $(x,y) \in R_2$ and $(y,z) \in R_2$ this means
$x-y = m \in \mathbb{Z}$ and $y-z = n \in \mathbb{Z}$
then $x-z = (x-y)+(y-z) = m+n$ $\mathbb{Z}$
and $(x,z) \in R_2$

Lastly R_2 is symmetric because $(x,y) \in R_2$ means $x-y = n \in \mathbb{Z}$, then $y-x = -n \in \mathbb{Z}$ and $(y,x) \in R_2$.

Therefore R_2 is an equivalence relation.

If R is an equivalence relation on A, then for an element $a \in A$ the <u>equivalence class</u> of a is

$$[a] = [a]_R = \{ b : (a,b) \in R \}$$

Consider the equivalence relation on $\mathbb{Z}$

$$R = \{ (a,b) : a+b \text{ is even} \}$$

1) Prove R is actually an equivalence relation.

2) Try to find the equivalence classes.

$\boxed{\text{Theorem}}$ A set, R equivalence relation

The following are equivalent

(i) $(a,b) \in R$ (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Pf) Let's show (i) $\Rightarrow$ (ii)

Assume $(a,b) \in R$. Take $c \in A$ so that $c \in [a]$, this means $(a,c) \in R$. Also means $(b,c) \in R$ since $(b,a) \in R$ and $(a,c) \in R$ and R is transitive. This shows $[a] \subseteq [b]$

Similarly if $c \in [b]$ then $(c,b) \in R$ so $(a,c) \in R$ also because

$(a,b) \in R$, $(b,c) \in R$ and R is transitive

Also need to prove others...

$\{X_i\}$ partition a set X if

(1.) $\bigcup_i X_i = X$ (2.) $X_i \cap X_j = \emptyset$ for $i \neq j$

$\boxed{\text{Theorem}}$ If R is an equivalence relation on A, then the equivalence classes partition A.