5.3 Recursive Definitions

Σ finite set

Σ^* set of strings over Σ

(1) $\lambda \in \Sigma^*$ (λ denotes empty string)

(2) $w \in \Sigma^*$ and $x \in \Sigma$ implies $wx \in \Sigma^*$

Example $\Sigma = \{a, b\}$

$\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

The length of a string $w \in \Sigma^*$ is denote $\ell(w)$ and defined by

$$\ell(\lambda) = 0$$

$$\ell(wx) = \ell(w) + 1 \quad \text{if } w \in \Sigma^*, x \in \Sigma$$

Example $\ell(a) = 1$ $\quad \ell(a) = \ell(\lambda a) = \ell(\lambda) + 1$

$\ell(ab) = 2$ $\quad \ell(ab) = \ell(a) + 1$

$= \ell(\lambda) + 1 + 1$

$= 2$

For $x, y \in \Sigma^*$ let xy denote concatenation

$x = aab, y = aba$ $\quad xy = aababa$

We will now consider <u>structural induction</u>

(1) <u>Base case</u>: Show the result holds for all base cases of recursive definition

(2) <u>Recursive step</u>: Show the result is true if it is true for each element used in recursion.

<u>Example</u>: Prove that $l(xy) = l(x) + l(y)$ for any $x, y \in \Sigma^*$

Pf) Take any $x \in \Sigma^*$.

(1) We will look at $y = \lambda$.

LHS $l(xy) = l(x\lambda) = l(x)$

RHS $l(x) + l(\lambda) = l(x) + 0 = l(x)$

(2) Assume that $l(xy) = l(x) + l(y)$ for some $y \in \Sigma^*$. Consider some $a \in \Sigma$.

LHS $l(xya) = l(xy) + 1 = l(x) + l(y) + 1$

RHS $l(x) + l(ya) = l(x) + l(y) + 1$

Ackermann's functions

Input $(m,n) \in \mathbb{N} \times \mathbb{N}$

$$A(m,n) = \begin{cases} 2n & m = 0 \\ 0 & m \geq 1, n = 0 \\ 2 & m \geq 1, n = 1 \\ A(m-1, A(m, n-1)) & m \geq 1, n \geq 2 \end{cases}$$

$A(0,0) = 0$

$A(1,0) = 0$

$A(0,1) = 2$

$A(1,1) = 2$

$A(1,2) = A(0, A(1,1)) = A(0,2) = 4$

$\cancel{A(2,2) = A(1, A(2,1)) = A(1,2) = A(0, A(0,3))}$

$= A$

$A(2,2) = A(1, A(2,1))$ $\quad A(2,1) = 2$

$= A(1,2)$

$= 4$

Prove $A(m,2)=4 \quad \forall\, m \geq 1$

Prove $A(1,n)=2^n \quad \forall\, n \geq 1$

Pf) Assume $A(m,2)=4$ for some $m \geq 1$. Notice $A(1,2)=4$ is the base case which we have checked.

$$A(m+1,2) = A(m, A(m+1, 1))$$
$$= A(m,2)$$
$$= 4$$

For a base case we see

$$A(1,1) = 2 = 2^1.$$

Now assume $A(1,n)=2^n$ ~~then~~ for $n \geq 1$,

$n \geq 1$
$n+1 \geq 2$

$$A(1,n+1) = A(0, A(1,n))$$
$$= A(0, 2^n)$$
$$= 2 \cdot 2^n$$
$$= 2^{n+1}$$