8.2 Solving Linear Recurrence Relations

A linear homogeneous recurrence relation of degree k with constant coefficients is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + a_k a_{n-k}$$

$$c_1, c_2, \dots, c_k \in \mathbb{R} \text{ and } c_k \neq 0$$

$a_n = a_{n-1} + a_{n-2}$ Yes

$a_n = 5a_{n-1} + 4a_{n-2} + 6a_{n-3}$ Yes

$a_n = a_{n-2}^2 + a_{n-5}$ No

$a_n = n \cdot a_{n-2}$ No

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + a_k a_{n-k}$$

$\downarrow$

characteristic equation

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

equivalent to

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

Solve the polynomial equation given by the characteristic equation gives a solution to the recurrence

If $r_1, r_2, \ldots, r_k$ are distinct roots, then

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r^k$$

for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$

Let's try degree two

$$a_n = a_{n-1} + a_{n-2} \qquad a_0 = 0, \; a_1 = 1$$

$$r^2 - r - 1 = 0 \qquad r = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$a_0 = \alpha_1 + \alpha_2 = 0$$

$$a_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1}{2}(\alpha_1 + \alpha_2) + \frac{1}{2}(\alpha_1 - \alpha_2)\sqrt{}$$

$$= 1$$

$$\alpha_2 = -\alpha_1$$

$$\alpha_1 \sqrt{5} = 1 \qquad \alpha_1 = \frac{1}{\sqrt{5}}$$

$$\alpha_2 = \frac{-1}{\sqrt{5}}$$

$a_n = a_{n-1} + a_{n-2} \qquad a_0 = 0,\ a_1 = 1$

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$0, 1, 1, 2, 3, 5, 8, \dots$

Try solving

$b_n = -b_{n-1} + 6b_{n-2} \qquad b_0 = 0,\ b_1 = 2$

$r^2 + r - 6 = (r-2)(r+3)$

$r_1 = 2,\ r_2 = -3$

$b_n = \alpha_1 2^n + \alpha_2 (-3)^n$

$b_0 = \alpha_1 + \alpha_2 = 0 \qquad \alpha_2 = -\alpha_1$

$b_1 = 2\alpha_1 - 3\alpha_2 = 2$

$5\alpha_1 = 2 \qquad \alpha_1 = 2/5,\ \alpha_2 = -2/5$

$b_n = \left(\frac{2}{5}\right)(2^n) + \left(\frac{-2}{5}\right)(-3)^n$

$0, 2, -2, 14, \dots$

8.3 Divide-and-Conquer

Find the maximum value of the sequence

1, 3, 2, 7, 10, 5, 4, 12, 6, 3, 1.

Scan across and find 12

Alternatively split in half

1, 3, 2, 7, 10 $\rightsquigarrow$ 10

5, 4, 12, 6, 3, 1 $\rightsquigarrow$ 12

$\rightsquigarrow$ 12

Sort the sequence

10, 6, 5, 17, 11, 3

10, 6, 5	17, 11, 3
↓	↓
5, 6, 10	3, 11, 17
↓	↓
5, 6, 10	11, 17
↓	↓
6, 10	11, 17
↓	↓
10	11, 17
↓	↓
∅	11, 17

3, 5, 6, 10, 11, 17

Let $f(n)$ denote the time required to sort n numbers with the "split in half & merge" method. (Called Merge sort)

$$f(n) = 2f\left(\frac{n}{2}\right) + n$$

$f\left(\frac{n}{2}\right)$ time required to sort each half
$2f\left(\frac{n}{2}\right)$ time required to sort both halves
n comparisons required to merge

$2f\left(\frac{n}{2}\right) + n$ in total

Let $g(n)$ denote the time required to find maximum value of sequence by splitting in half. Write a recurrence for $g(n)$

$$\boxed{\begin{array}{l} g(n) = 2g\left(\frac{n}{2}\right) + 1 \\ 2g\left(\frac{n}{2}\right) \text{ finds maximum of each half} \\ 1 \text{ additional step to compare maximums} \end{array}}$$

Master Theorem

Let f be an increasing function such that

$$f(n) = a f\left(\frac{n}{b}\right) + cn^d$$

then $f(n)$ is

$$\begin{cases} O(n^d) & a < b^d \\ O(n^d \log n) & a = b^d \\ O(n^{\log_b a}) & a > b^d \end{cases}$$

Take $b=2$, $n=8$, $d=1$

$$\begin{aligned} f(8) &= a f(4) + c \cdot 8 \\ &= a(af(2) + c \cdot 4) + c \cdot 8 \\ &= a(a(af(1) + c \cdot 2) + c \cdot 4) + c \cdot 8 \\ &= a^3 f(1) + a^2 \cdot c \cdot 2 + a \cdot c \cdot 4 + c \cdot 8 \end{aligned}$$

Assume $a=b^d$, n is a power of b $n=b^k$

$$f(n)=b^d\cdot f\left(\frac{n}{b}\right)+c\cdot n^d$$

$$f(b^k)=b^d f(b^{k-1})+c\cdot b^{d\cdot k}$$

$$=b^d\left(b^d\left(f(b^{k-2})\right)+c\cdot b^{d(k-1)}\right)+c\cdot b^{d\cdot k}$$

$$=b^{kd}f(1)+\sum_{j=1}^{k} c\cdot b^{dk}$$

$$=b^{kd}f(1)+k\cdot c\cdot b^{dk}$$

$$=n^d f(1)+\log_b(n)\cdot c\cdot n^d$$

$$O(\log_b(n)\cdot n^d)$$

$x\in\mathbb{R},\ n\in\mathbb{Z}^+$

$x^n=\underbrace{x\cdot x\cdots x}_{n\text{-times}}$ can be computed as

$$x,\ x\cdot x=x^2,\ x^2\cdot x=x^3,\ \dots,\ x^{n-1}\cdot x=x^n$$

requiring $n-1$ multiplications (i.e. $O(n)$)

Can you do better?

$$x^n=(x^{n/2})\cdot(x^{n/2})$$

$f(n)=1f\left(\frac{n}{2}\right)+2$ $\qquad O(\log(n))$

$a=1$ $c=2$

$b=2$ $d=0$