If A, B, and C are sets and $f: A \to B$ and $g: B \to C$ are, functions then

$$g \circ f: A \to C$$

defined for $(g \circ f)(a) = g(f(a))$ for all $a \in A$ is the composition of f and g.

If $f: A \to B$ and $g: B \to A$, then we get a function

$$g \circ f: A \to A$$

And when $(g \circ f)(a) = a$ for all $a \in A$, we say g is the inverse of f and writing $g = f^{-1}$.

$\boxed{\text{Thm}}$ A function f has a inverse iff f is a bijection.

Example $h: \mathbb{Z} \to \mathbb{Z}$ $\quad h(n) = n+2$

$h^{-1}: \mathbb{Z} \to \mathbb{Z}$ $\quad h^{-1}(n) = n-2$

Check take $n \in \mathbb{Z}$

$$\begin{aligned}(h^{-1} \circ h)(n) = h^{-1}(h(n)) &= h^{-1}(n+2) \\ &= (n+2)-2 \\ &= n\end{aligned}$$

Also $(h \circ h^{-1})(n) = h(h^{-1}(n)) = h(n-2) = (n-2)+2 = n$

Alternative proof that a function is bijective, exhibit an inverse function (and prove it is the inverse).

Show $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x+1$ is a bijection.

$y = 2x+1$

$y-1 = 2x$

$\frac{y-1}{2} = x$

$g: \mathbb{R} \to \mathbb{R}$

$g(x) = \frac{x-1}{2}$

$(g \circ f)(x) = g(2x+1) = \frac{(2x+1)-1}{2} = x$

$(f \circ g)(x) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x$

So $g = f^{-1}$ and f is a bijection

Consider $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = (x-5)^2$

Is f bijective? Can you restrict domain and codomain to make it bijective?

Let A, B, and C be sets. Let $f: A \to B$ and $g: B \to C$ be functions.

1) Show if g is not onto, then $g \circ f$ is not onto.

2) Show if $g \circ f$ is onto, then g is onto

2) is contrapositive of 1) so we only need to prove 1)

If g is not onto, then $\exists\, c \in C$ so that $g(b) \neq c \;\; \forall\, b \in B$.

Then $(g \circ f)(a) = g(f(a)) \neq c$ for any $a \in A$ and $g \circ f$ is not onto.

Give an example where $g \circ f$ is onto but f is not onto.

The <u>floor function</u> is a function from $\mathbb{R}$ to $\mathbb{Z}$ which assigns each real number the largest integer less than or equal to it. ~~We den~~ denot

The <u>ceiling function</u> is a function from $\mathbb{R}$ to $\mathbb{Z}$ which assigns each real number the smallest integer greater than or equal to it.

$\lfloor x \rfloor$ floor

$\lceil x \rceil$ ceiling

$\lceil 1 \rceil =$

$\lfloor 1 \rfloor =$

$\lceil 1.5 \rceil =$

$\lfloor 2.7 \rceil =$

$\lfloor \pi \rfloor =$

$\lceil 10/3 \rceil =$

Find formulas for $\lfloor -x \rfloor$ and $\lceil -x \rceil$ in terms of $\lfloor x \rfloor$ and $\lceil x \rceil$ (no negatives inside)

$\lfloor -x \rfloor = -\lceil x \rceil$ $\qquad$ $\lceil -x \rceil = -\lfloor x \rfloor$

Let's prove $\lfloor -x \rfloor = -\lceil x \rceil$

If $\lfloor -x \rfloor = n \in \mathbb{Z}$, this means

$$n \leq -x < n+1$$
$$-n \geq x > -n-1$$

$$-n-1 < x \leq -n$$

So $\lceil x \rceil = -n$ and $\lfloor -x \rfloor = -\lceil x \rceil$.

$x \in \mathbb{R}$ $\quad$ $n \in \mathbb{Z}$

$\lfloor x \rfloor = n$ iff $n \leq x < n+1$

$\lceil x \rceil = n$ iff $n-1 < x \leq n$

$\lfloor x \rfloor = n$ iff $x-1 < n \leq x$

$\lceil x \rceil = n$ iff $x \leq n < x+1$

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

When does $\lfloor x \rfloor = \lceil x \rceil$?

Compute $\lfloor x+n \rfloor$ and $\lceil x+n \rceil$ for $n \in \mathbb{Z}$.

1) Let $f: A \to A$, $g: A \to B$, and $h: B \to A$ be functions for sets A, B.

How can the functions f, g, and h be composed?

2) Prove $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$ for any $x \in \mathbb{R}$.

3) Determine if $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto

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$f(a,b) = b$

$f(a,b) = a+b$

$f(a,b) = |a|$

$f(a,b) = 3a+6b$